Last time: $a, b \in \mathbb{C}$, $a \neq 0$. $a^b = e^{b\log(a)}$ where some branch of log is chosen

Ex: Pick $[0, 2\pi)$ as our branch of log.

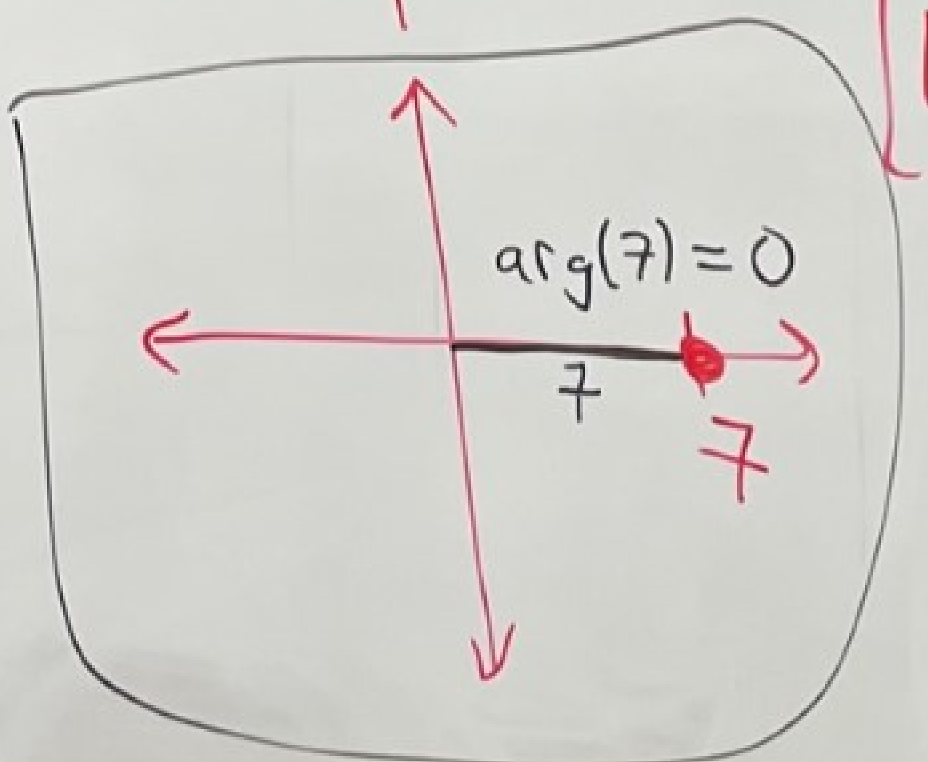
Then,

$$7^3 = e^{3\log(7)} = e^{3[\ln(7) + i0]} = e^{\overset{\text{real \#}}{3\ln(7)}} = e^{\ln(7^3)} = 7^3 = 343$$

complex version of 7^3

normal 7^3

$$\boxed{\log(z) = \ln|z| + i\arg(z)}$$



Sometimes, this def extends the real analysis def of a^b

Why?

Suppose $a, b \in \mathbb{R}$, $a > 0$.
Pick any branch of log.

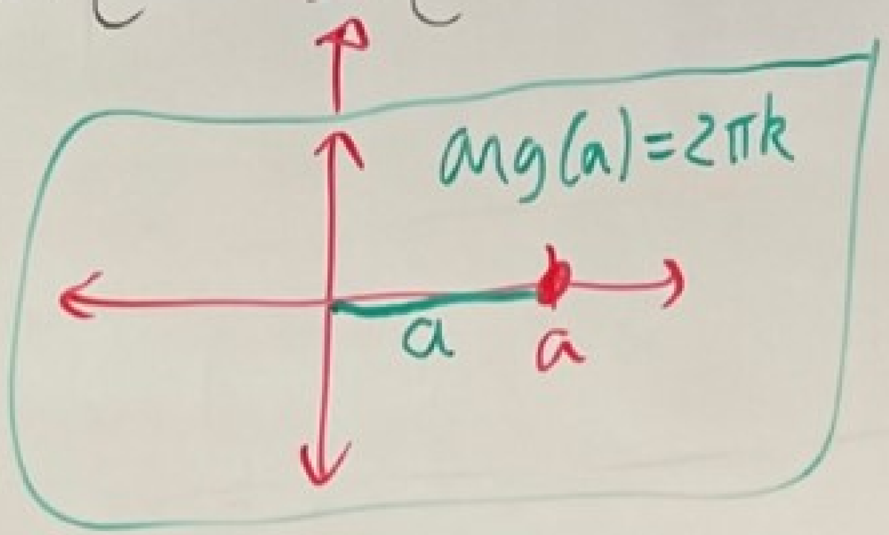
Then,

$$a^b = e^{b\log(a)} = e^{b[\ln(a) + i2\pi k]} = e^{b\ln(a) + ib2\pi k}$$

$$= \underbrace{\boxed{e^{b\ln(a)}}}_{\text{all real \# calculation}} e^{ib2\pi k}$$

$$= e^{\ln(a^b)} e^{ib2\pi k}$$

$$= \underbrace{a^b}_{\text{real \# } a^b} e^{ib2\pi k}$$



Note $e^{i2\pi bk} = 1$ iff $bk \in \mathbb{Z}$

So for example if you pick $[0, 2\pi)$ as your branch of log, then $k = 0$. So $bk = 0$.

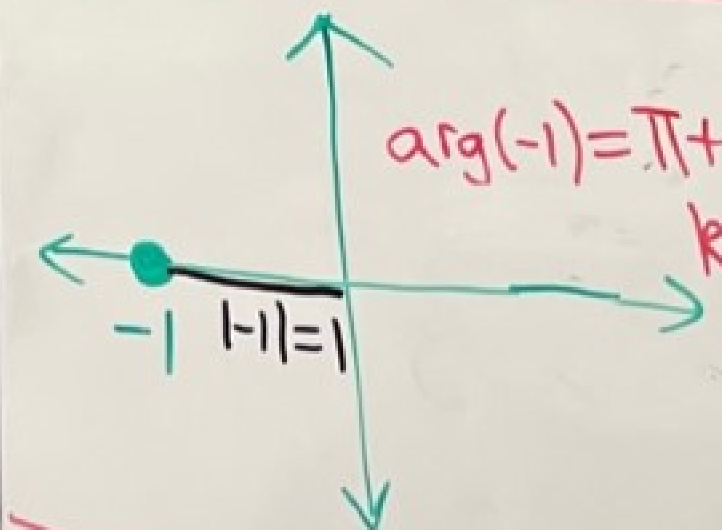
and $\underbrace{a^b}_{\text{complex } a^b} = a^b e^{i(2\pi bk)} = a^b e^{i0} = a^b \cdot 1 = \underbrace{a^b}_{\text{real \# } a^b}$

Ex: Let's calculate $(-1)^{1/2}$

For now let's wait to pick the branch of log.

$$(-1)^{1/2} = e^{\frac{1}{2}\log(-1)} = e^{\frac{1}{2}[\ln|-1| + i\arg(-1)]} = e^{\frac{1}{2}[\underbrace{\ln(1)}_{0} + i(\pi + 2\pi k)]} = e^{i(\frac{\pi}{2}+\pi k)} = \begin{cases} i, & k \text{ is even} \\ -i, & k \text{ is odd} \end{cases}$$

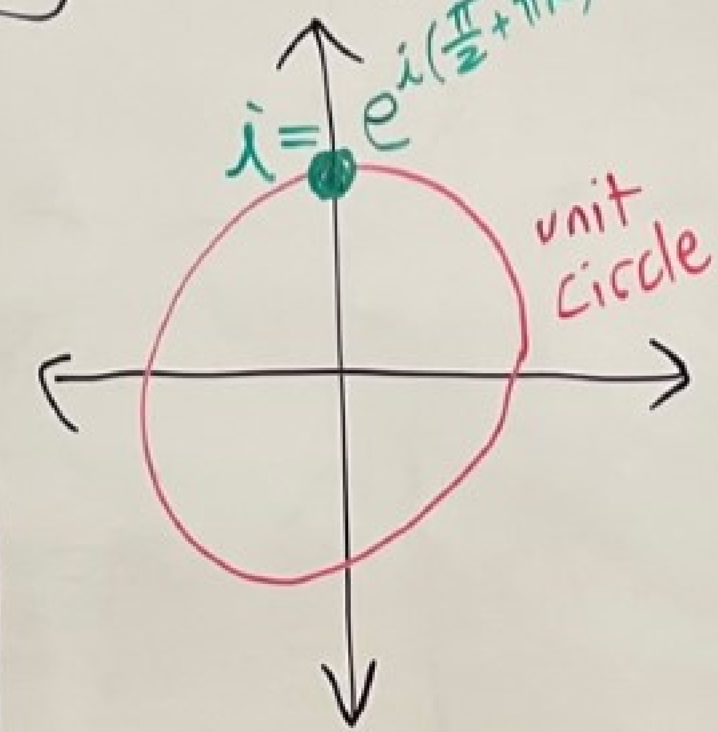
instead of pic could use
$e^{i(\frac{\pi}{2}+\pi k)} = \cos(\frac{\pi}{2}+\pi k) + i\sin(\frac{\pi}{2}+\pi k)$



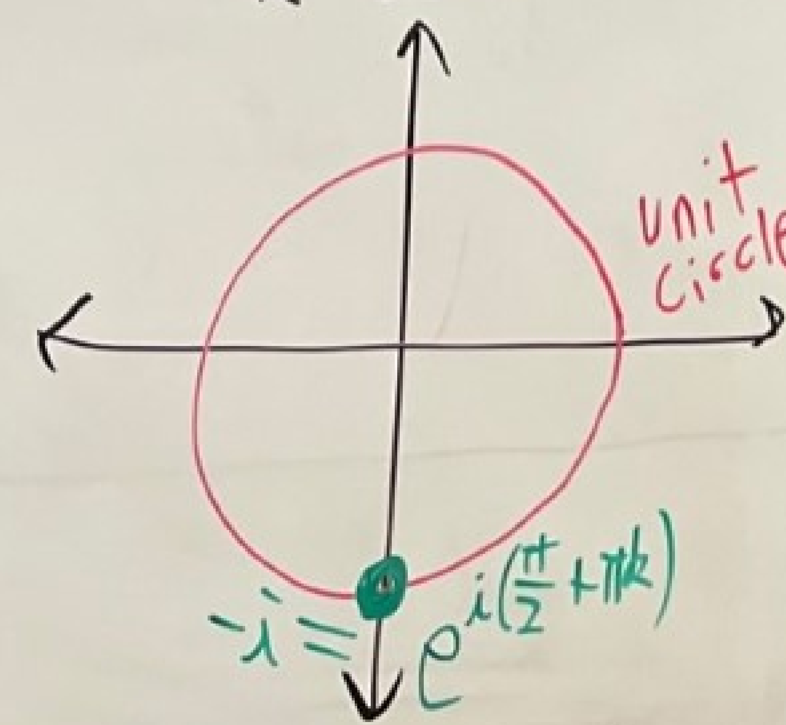
$\arg(-1) = \pi + 2\pi k$
$k \in \mathbb{Z}$

So the answer depends on the branch of log you pick.

k even



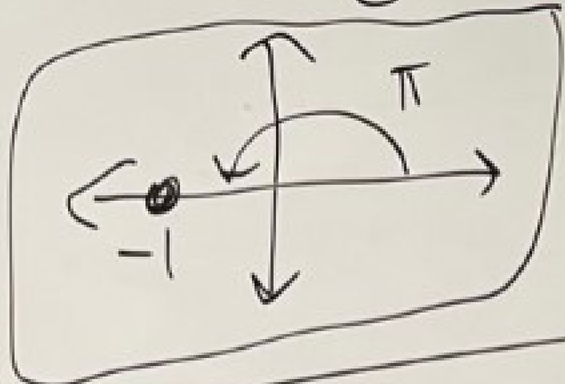
k odd



Ex: Pick $[0, 2\pi)$ as our branch of log.

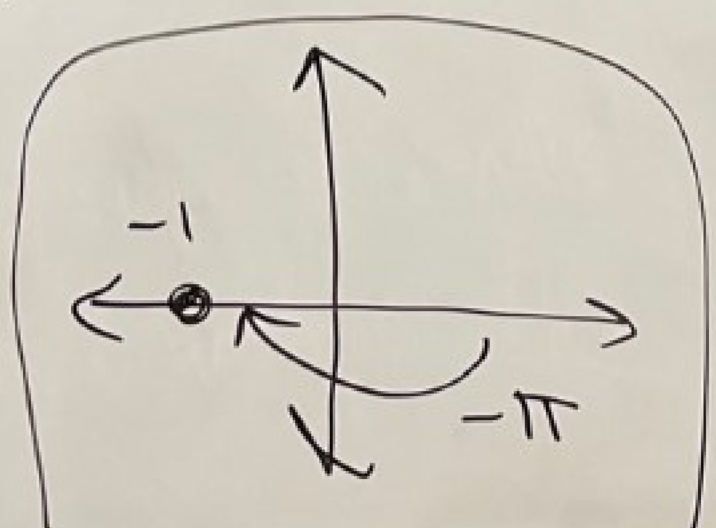
Then, $(-1)^{1/2} = e^{\frac{1}{2}\log(-1)} = e^{\frac{1}{2}[\underbrace{\ln(1)}_{0} + i\pi]} = e^{i\frac{\pi}{2}} = i$

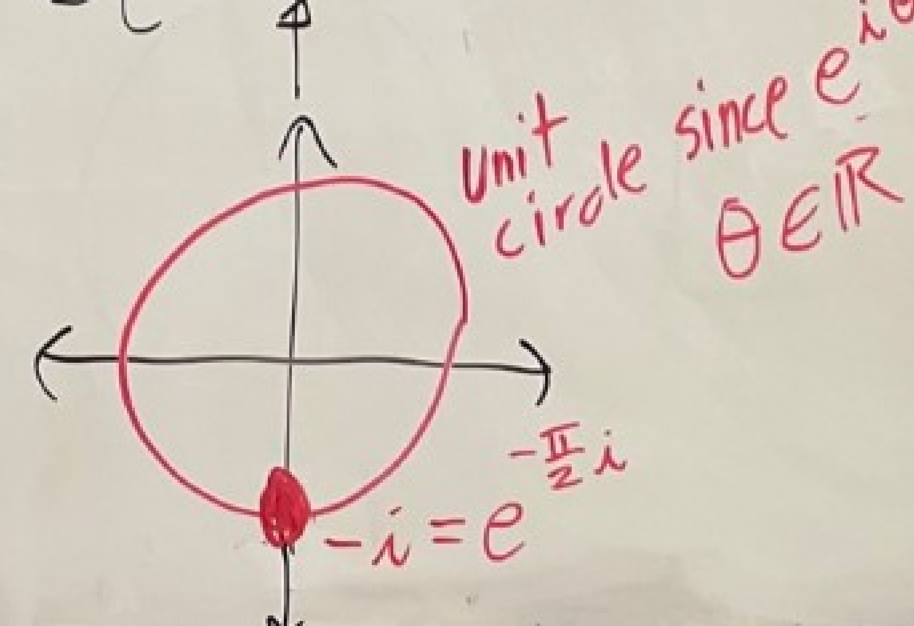
$$\boxed{e^{i\frac{\pi}{2}} = \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right) = 0 + i = i}$$



Ex: Pick $[-\pi, \pi)$ as our branch of log.

Then, $(-1)^{1/2} = e^{\frac{1}{2}\log(-1)} = e^{\frac{1}{2}[\underbrace{\ln(1)}_{0} + i(-\pi)]} = e^{-\frac{\pi}{2}i} = -i$





Def: Let $n \geq 2$ be an integer.

Define an <u>n-th root function</u> by

$$\sqrt[n]{z} = z^{1/n} = e^{\frac{1}{n}\log(z)}$$

where a specific branch of log is chosen.

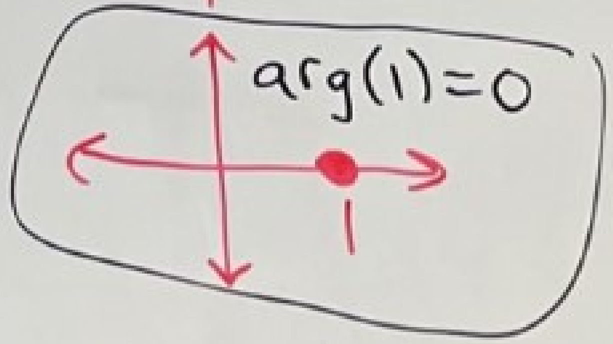
This is called picking a <u>branch of the n-th root function.</u>

Ex: Pick $[0, 2\pi)$ as our branch of log. Let $n = 2$.

Let $f(z) = z^{1/2} = e^{1/2 \log(z)}$

Then,

$$f(1) = e^{\frac{1}{2}\log(1)} = e^{\frac{1}{2}[\underbrace{\ln(1)}_{0} + i\,0]} = e^{0} = 1$$

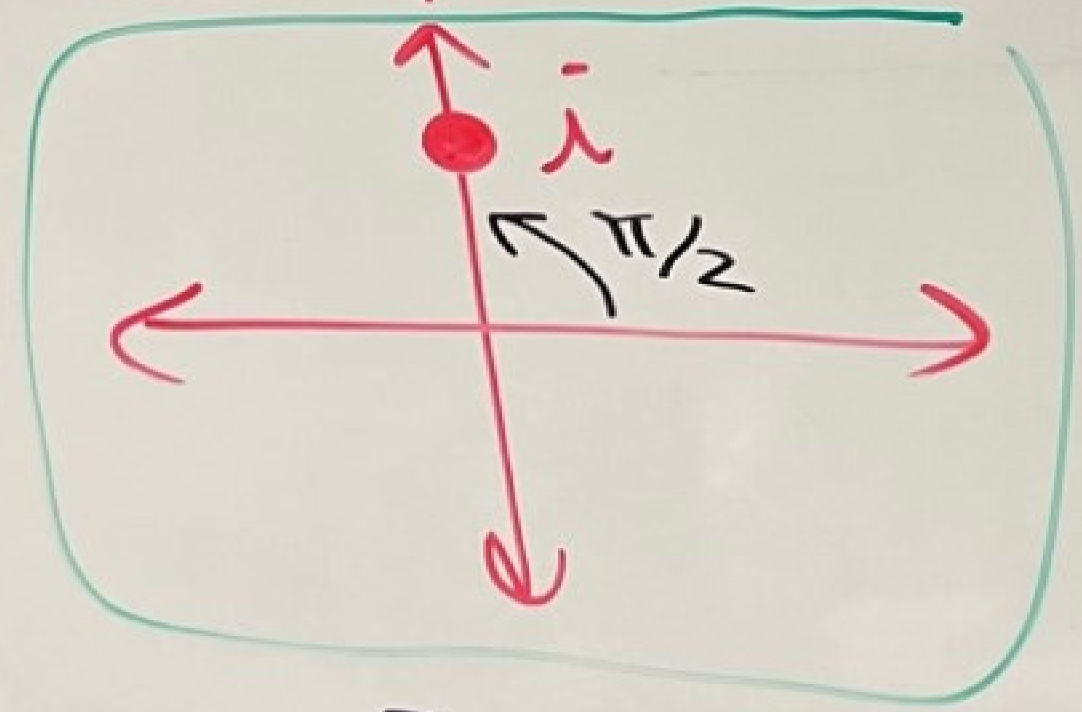


We did $f(-1)$ previously.

$f(-1) = i$

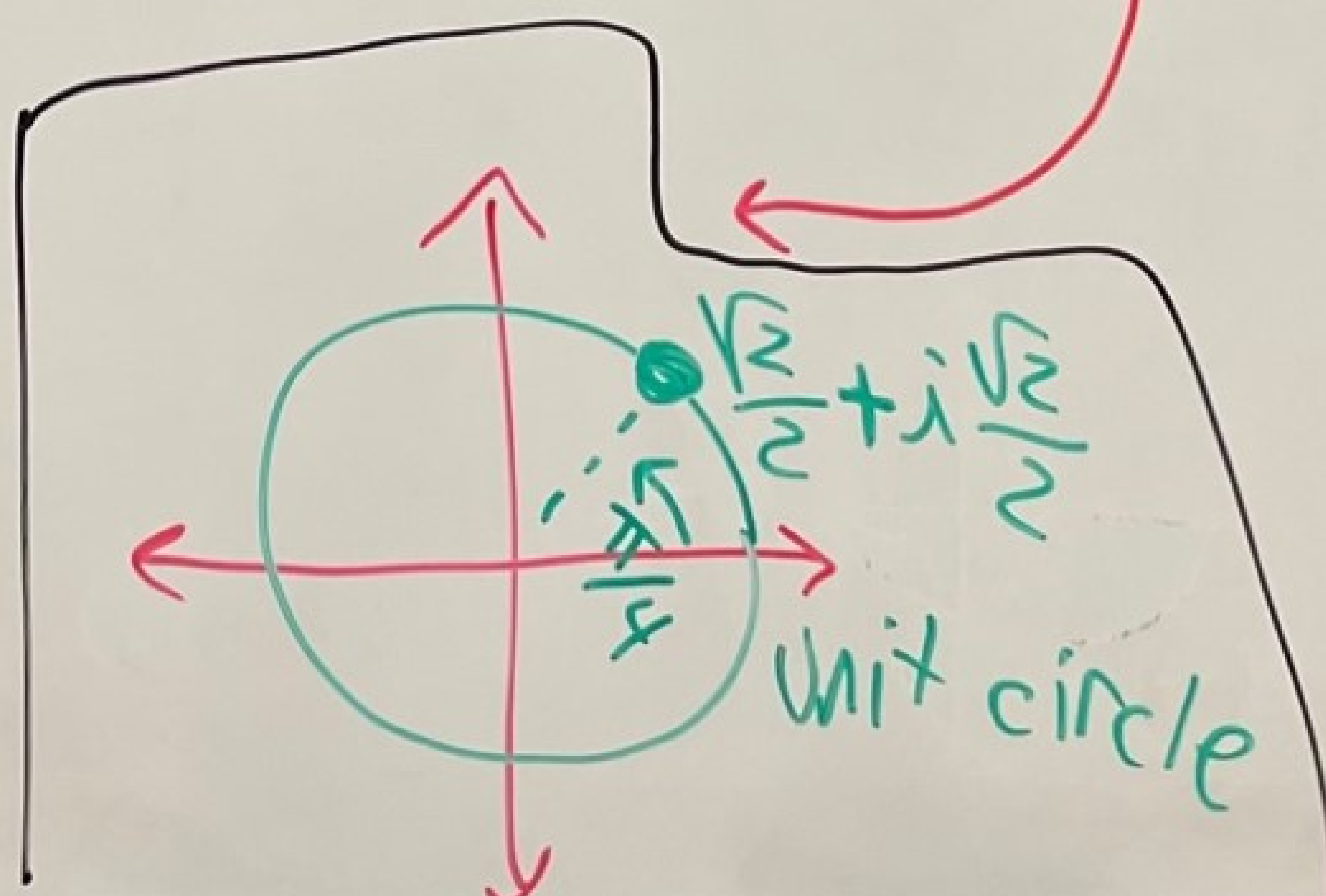
What is $f(i)$?

$$f(i) = e^{\frac{1}{2}\log(i)} = e^{\frac{1}{2}\left[\underbrace{\underbrace{\ln|i|}_{\ln(1)}}_{0} + i\arg(i)\right]}$$



$$= e^{\frac{1}{2}\left[0 + i\frac{\pi}{2}\right]} = e^{i\frac{\pi}{4}} = \cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)$$

$$= \boxed{\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}}$$



HW 3 – Some Topology

Def: Let $z_0 \in \mathbb{C}$ and $r \in \mathbb{R}$ where $r > 0$.

The set

$$D(z_0;r) = \{ z \mid |z - z_0| < r \}$$

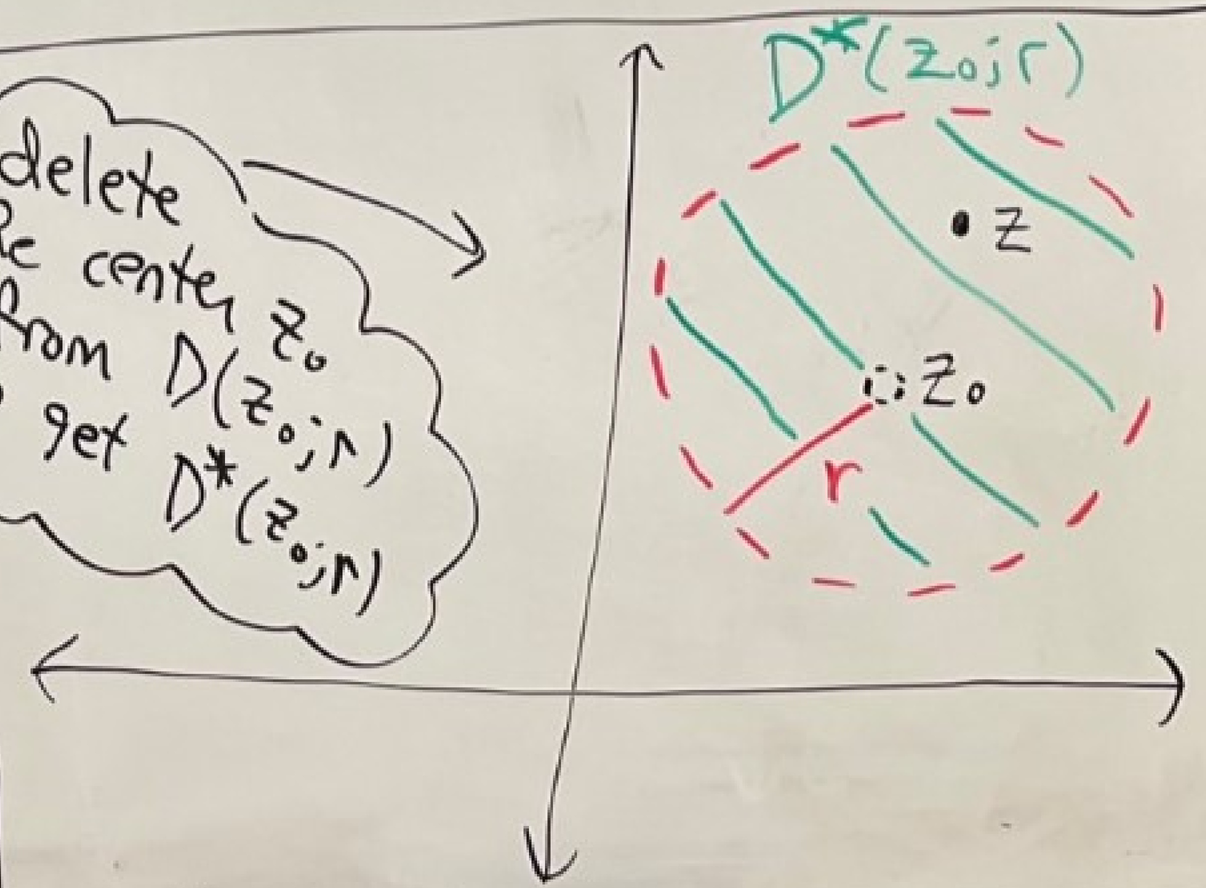
is called an <u>r-neighborhood of z_0</u>

The set

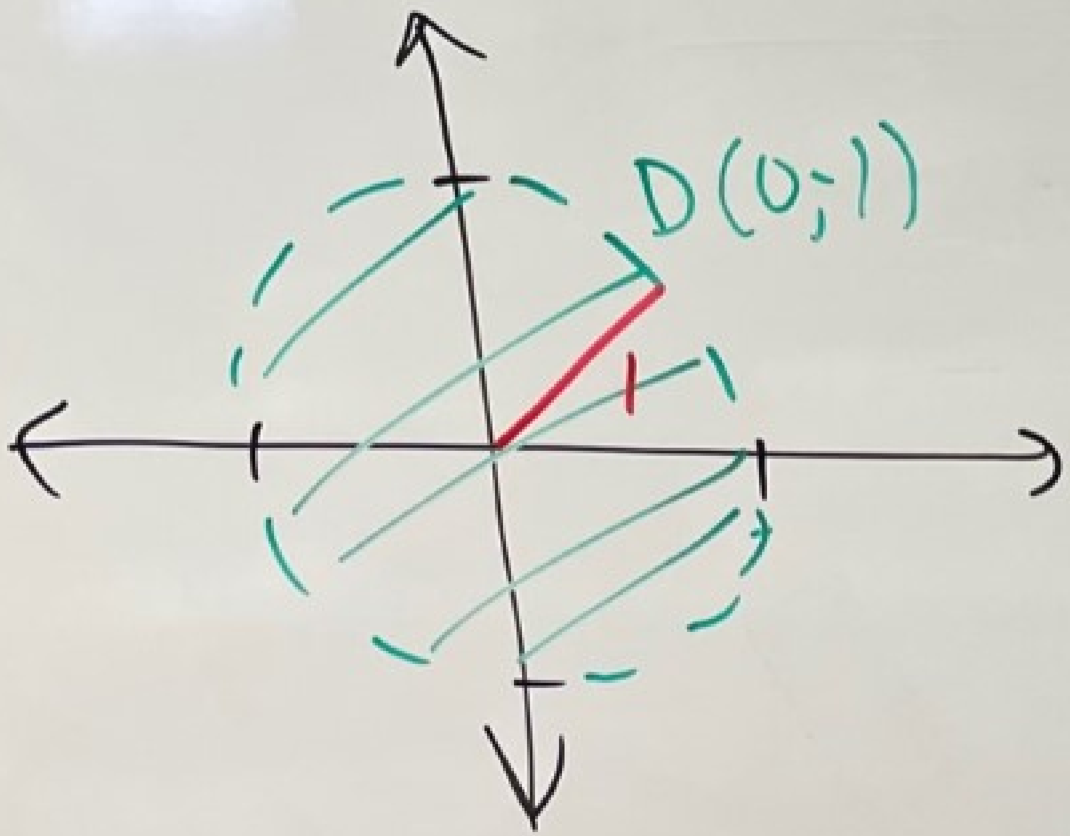
$$D^*(z_0;r) = D(z_0;r) - \{z_0\}$$
$$= \{ z \mid 0 < |z - z_0| < r \}$$

is called a <u>deleted r-neighborhood of z_0</u>

delete the center z_0 from $D(z_0;r)$ to get $D^*(z_0;r)$



Ex: $D(0;1)$



Ex: $D^*(5+5i;2)$