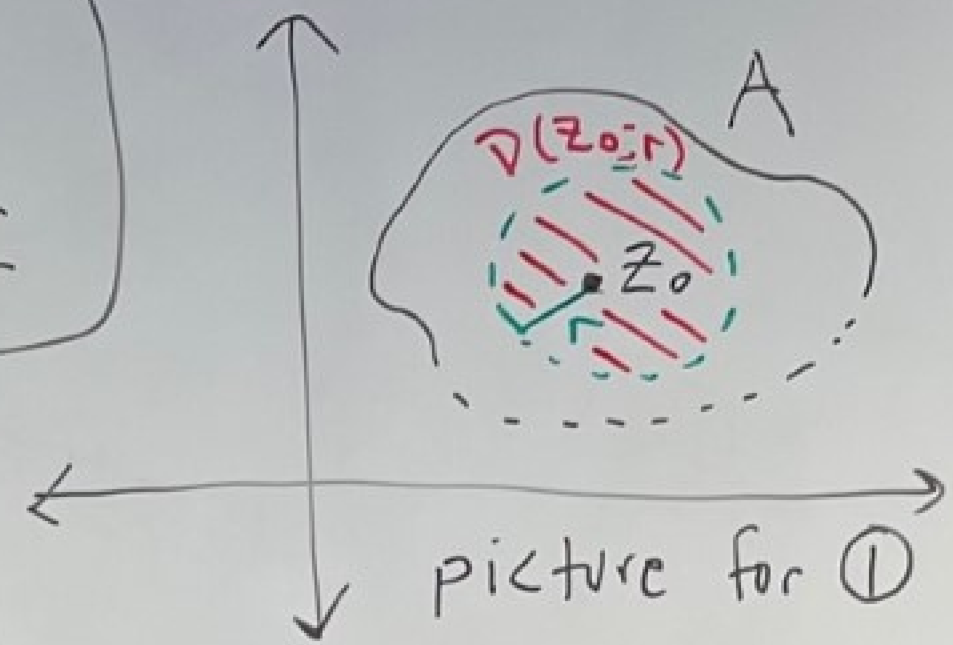
Def: Let $A \subseteq \mathbb{C}$.

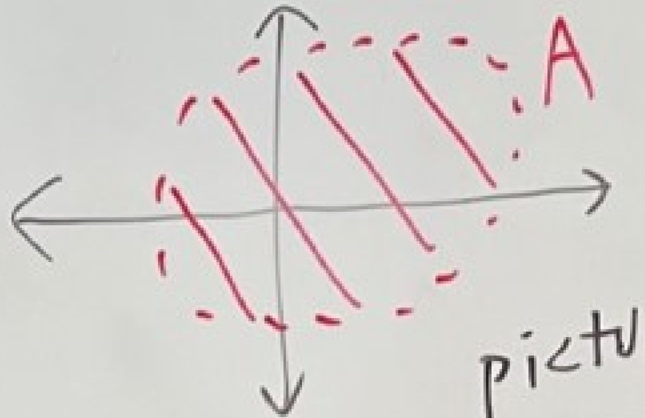
① Let $z_0 \in A$. We say that z_0 is an interior point of A if there exists $r > 0$ where $D(z_0; r) \subseteq A$.

② A is an open set if every element $z_0 \in A$ is an interior point.

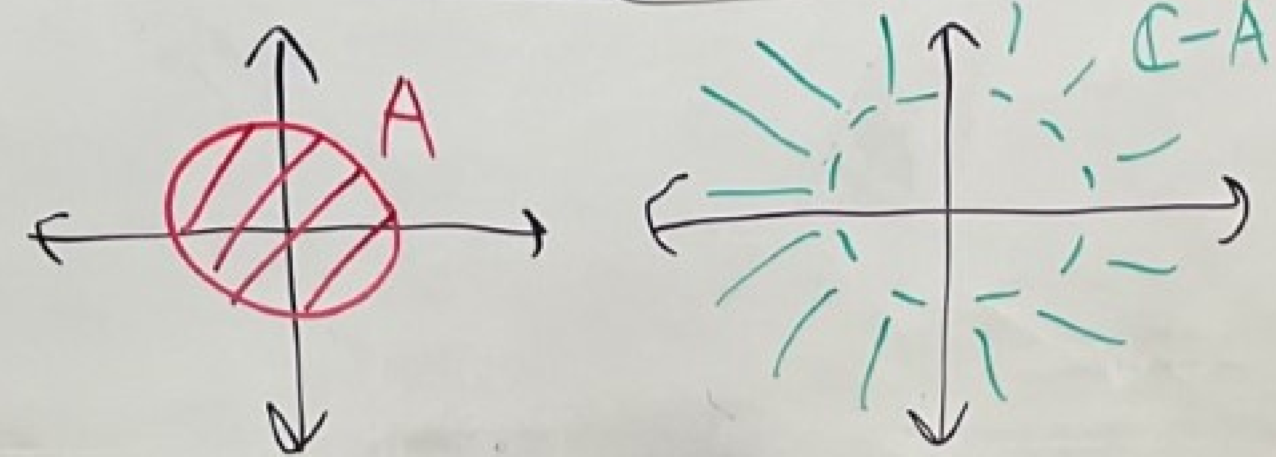
③ A is a closed set if the complement $\mathbb{C} - A$ is open.



picture for ①



picture of ②



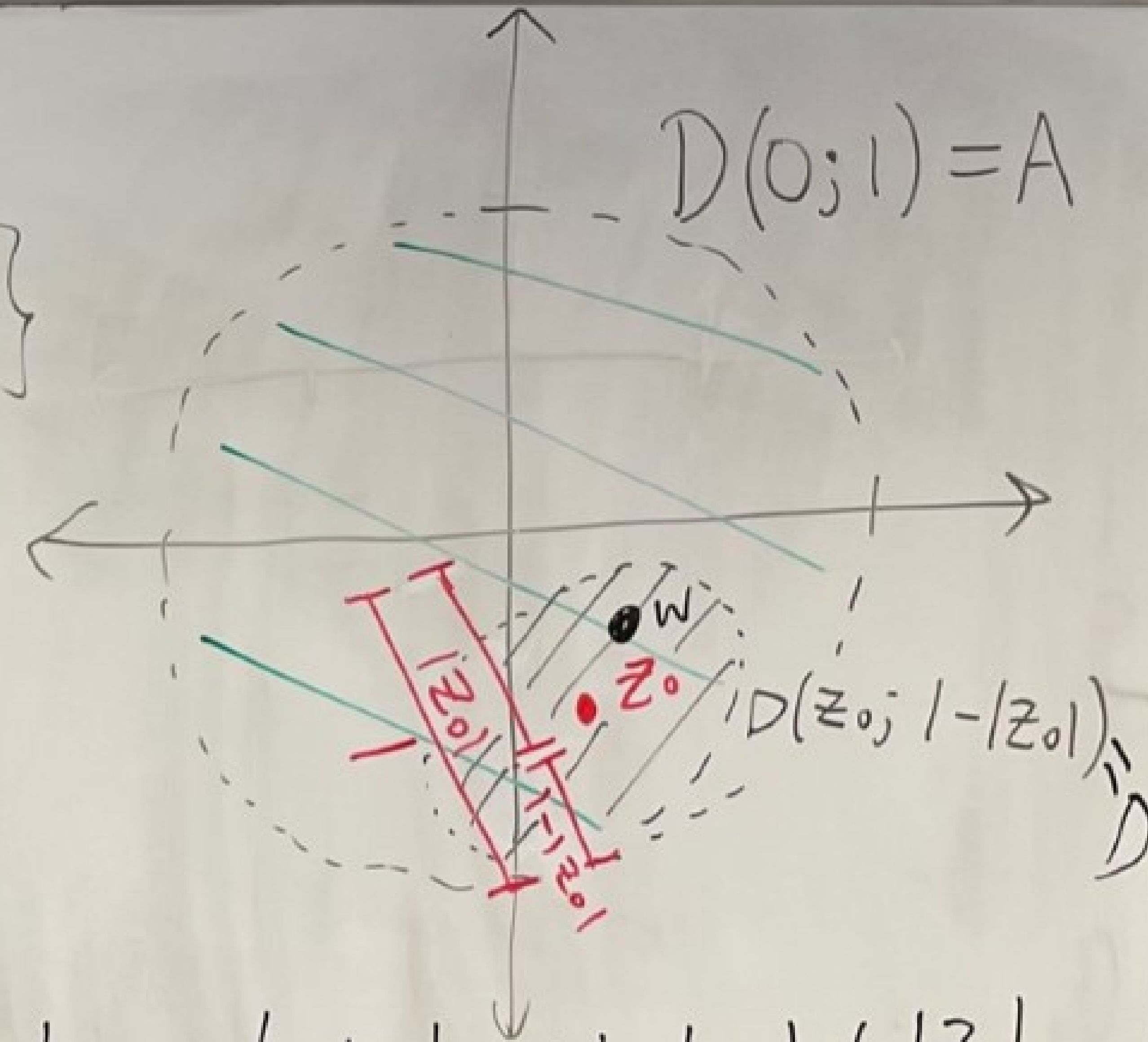
picture for ③

Ex: Let $A = D(0;1)$

$D(0;1) = \{z \mid |z-0| < 1\} = \{z \mid |z| < 1\}$

Let's show A is open.

Pick some $z_0 \in A$. We must show z_0 is an interior pt. of A



Let's show $D = D(z_0; 1-|z_0|) \subseteq A$.

Let $w \in D$.

Then, $|w - z_0| < 1 - |z_0|$.

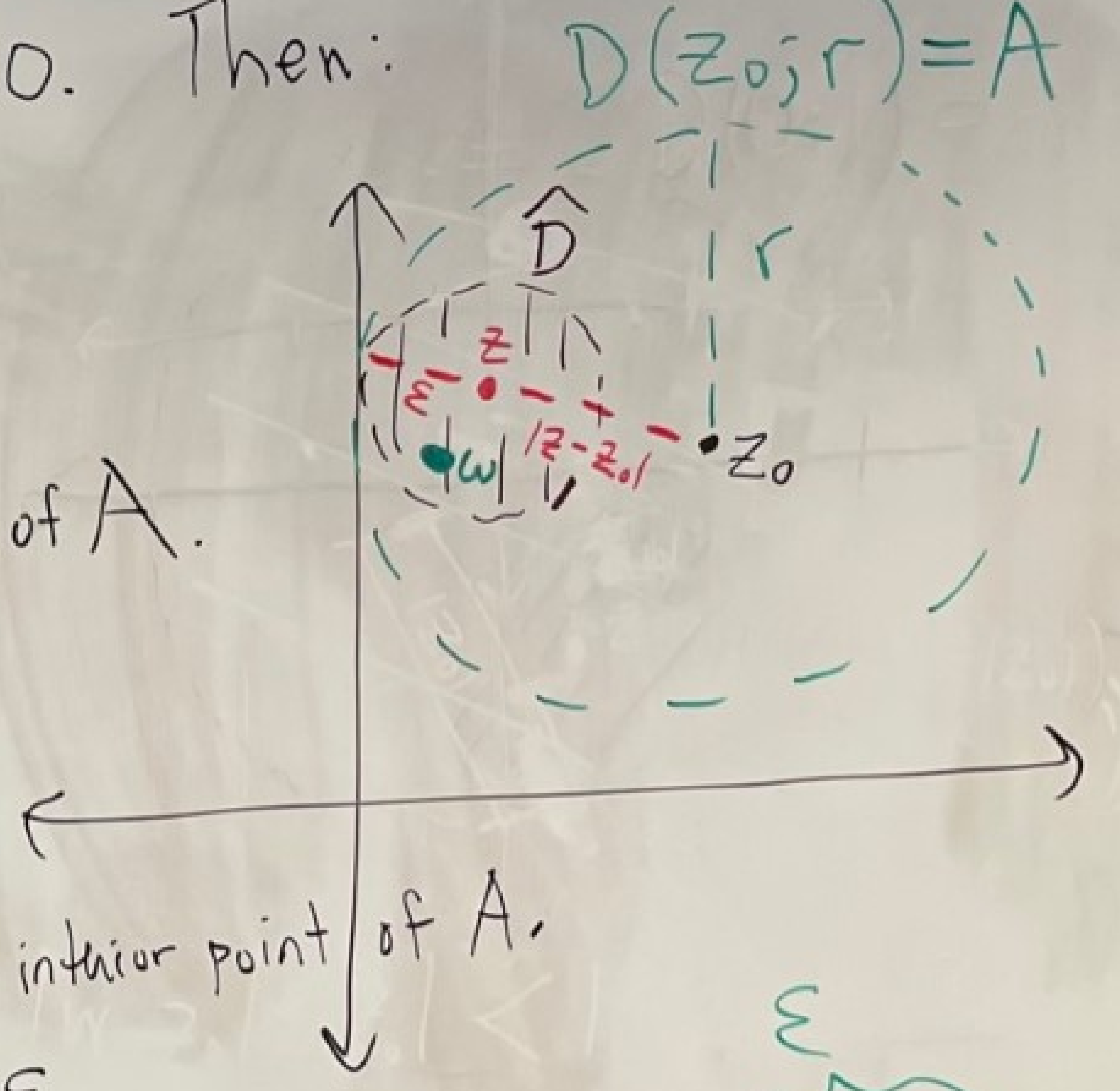
We have $|w-0| = |w| = |w - z_0 + z_0| \overset{\Delta}{\leq} |w - z_0| + |z_0| < \underbrace{1 - |z_0| + |z_0|}_{=1}$

So, $|w-0| = |w| < 1$

So, $w \in A$. Thus, $D \subseteq A$. So, z_0 is an interior pt. of A.

Thus, A is open. $\blacksquare$

Theorem: Let $z_0 \in \mathbb{C}$. Let $r \in \mathbb{R}$, $r > 0$. Then:



① $D(z_0;r) = \{z \mid |z-z_0| < r\}$ is open

② $D^*(z_0;r) = \{z \mid 0 < |z-z_0| < r\}$ is open.

proof of ①: Let $A = D(z_0;r)$

Let $z \in A$. We must show z is an interior pt. of A.

Let $\varepsilon = r - |z - z_0|$.

Since $z \in A$ we know $|z-z_0| < r$. So, $\varepsilon > 0$.

Let $\hat{D} = D(z;\varepsilon)$.

We need to show that $\hat{D} \subseteq A$, thus showing z is an interior point of A.

Let $w \in \hat{D}$. Since $w \in \hat{D}$ then $|w-z| < \varepsilon$.

We have $|w-z_0| = |w-z+z-z_0| \overset{\triangle}{\leq} |w-z| + |z-z_0| < \overbrace{\varepsilon}^{} + |z-z_0| = \underbrace{\overbrace{r-|z-z_0|}^{\varepsilon}+|z-z_0|}_{=r}$

So, $|w-z_0| < r$. So, $w \in A$. Thus, $\hat{D} \subseteq A$. So, z_0 is an int. pt. of A. So, A is open. $\boxed{/}$

② You try for fun

Ex: Let $B = \{ z \mid |z| \leq 1 \}$

Show B is not open.

To show B is not open we need to find a point in B that isn't an interior point.

Pick $z_0 = 1$.

$1 \in B$ but 1 isn't an interior point. Why?

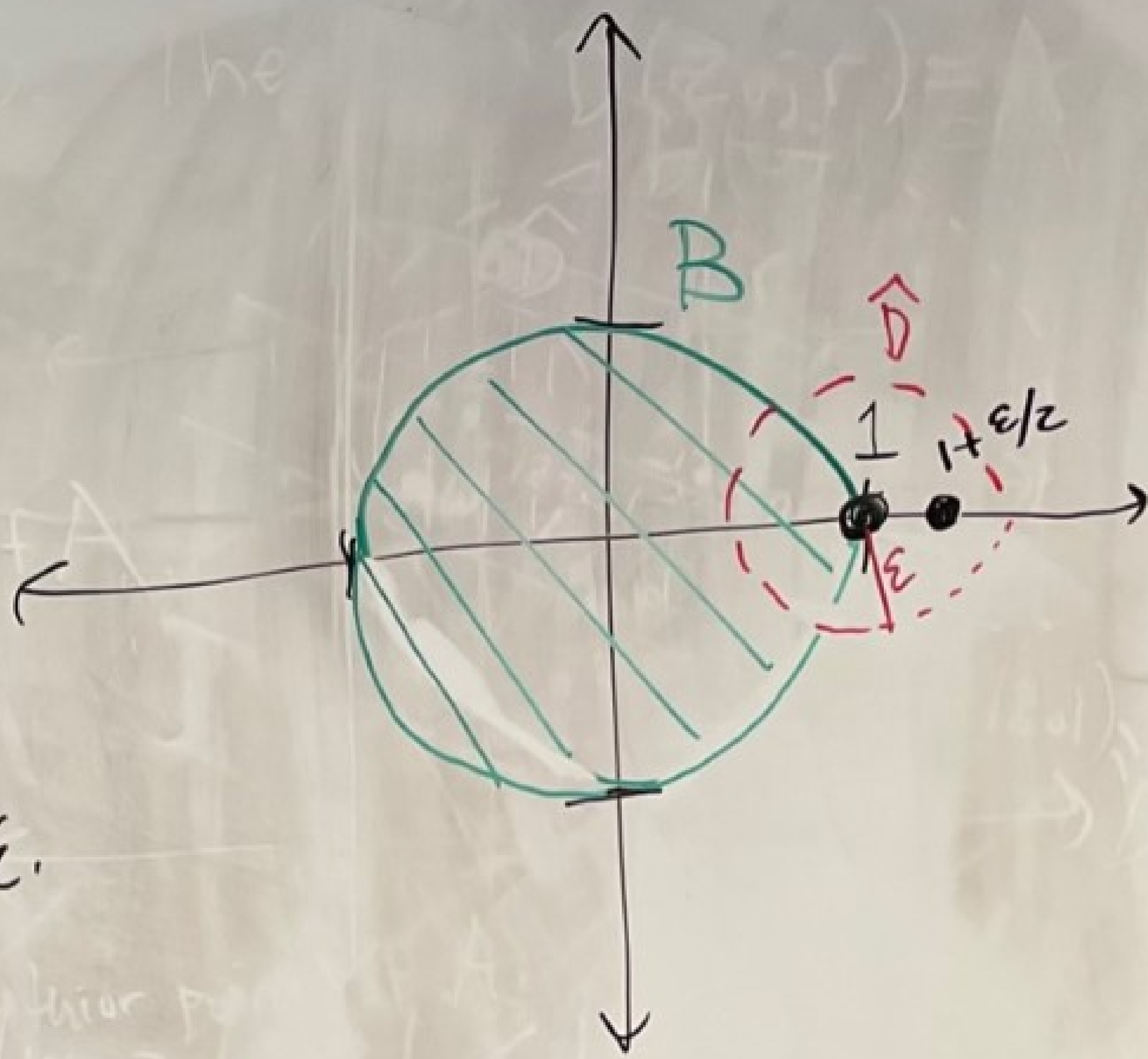
Suppose $\varepsilon > 0$. Let $\hat{D} = D(1; \varepsilon)$.

Then $1 + \frac{\varepsilon}{2} \in \hat{D}$ since $\left| \left(1 + \varepsilon/2\right) - 1 \right| = \left| \frac{\varepsilon}{2} \right| = \frac{\varepsilon}{2} < \varepsilon$.

But $\left| 1 + \frac{\varepsilon}{2} \right| = 1 + \frac{\varepsilon}{2} > 1$. So, $1 + \frac{\varepsilon}{2} \notin B$.

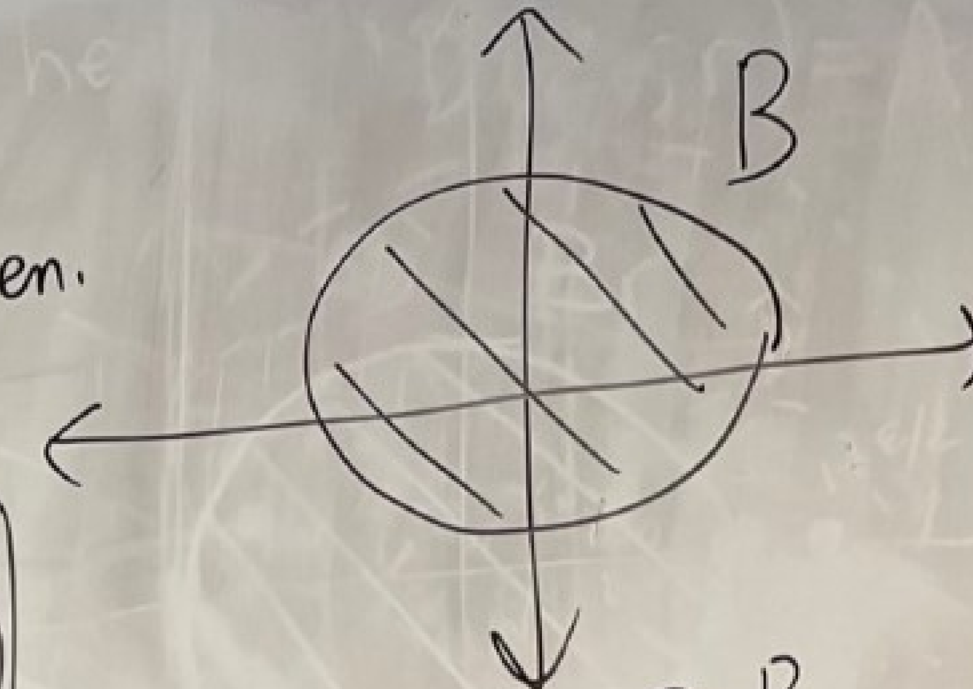
So, no matter what $\varepsilon > 0$ is, $\hat{D} = D(1; \varepsilon) \not\subseteq B$.

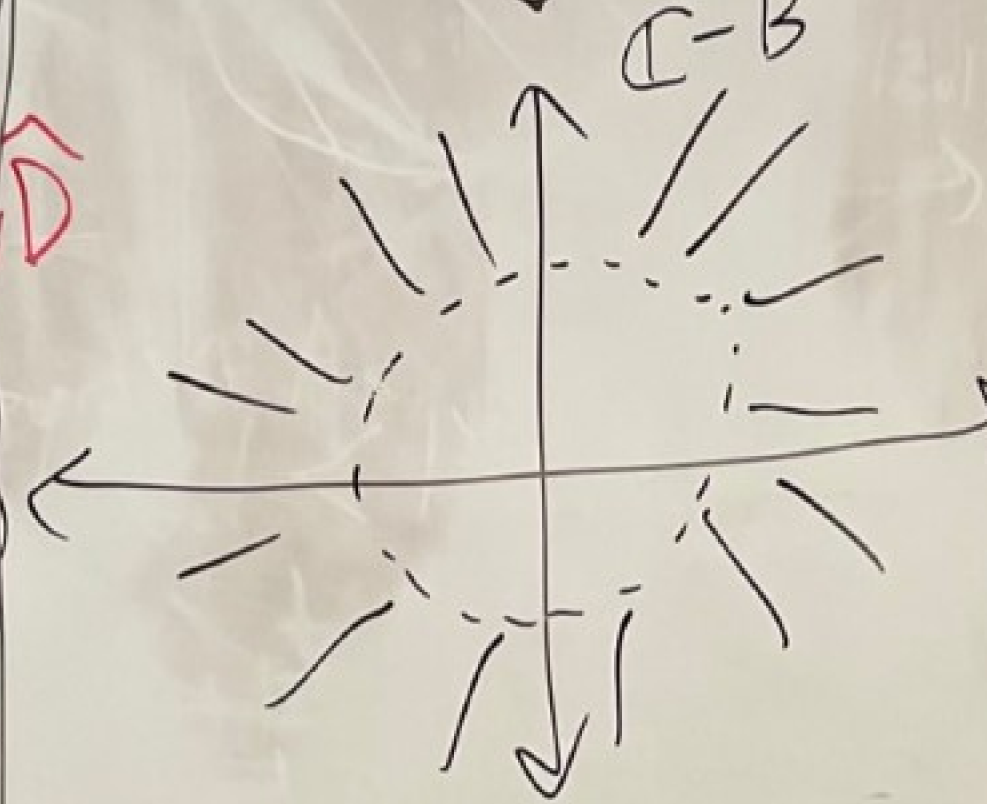
So, 1 cannot be an interior point of B. So, B is not open. ∎



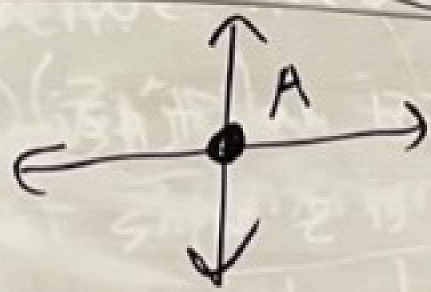
Note: $B=\{|z| \mid |z|\le 1\}$ is closed.

This is because $\mathbb{C}-B=\{z \mid |z|>1\}$ is open.





See HW 3 – #2(b)



Ex: Let $A=\{0\}$.

Let's prove A is closed.

We need to show $\mathbb{C}-A=\mathbb{C}-\{0\}$ is open.

Let $z_0\in\mathbb{C}-A$. So, $z_0\neq 0$.

Let $\hat{D}=D(z_0;|z_0|)$. Let's show $\hat{D}\subseteq\mathbb{C}-A$.

Let $w\in\hat{D}$. Let's show $w\neq 0$.

If $w=0$, then $|z_0|=|z_0-0|=|z_0-w| \underset{w\in\hat{D}}{<} |z_0|$

Then $|z_0|<|z_0|$. Contradiction.

So, $w\neq 0$ and so $w\in\mathbb{C}-A$.

Thus, $\hat{D}\subseteq\mathbb{C}-A$. So z_0 is an int. pt. of $\mathbb{C}-A$. So, $\mathbb{C}-A$ is open. So, A is closed. ∎