

Theorem: Suppose  $A \subseteq \mathbb{C}$  and  $z_0 \in \mathbb{C}$  and  $D^*(z_0; r) \subseteq A$  for some  $r > 0$ .

Suppose  $f(z) = f(x+iy) = u(x,y) + i v(x,y)$ .

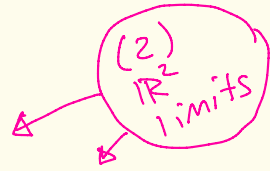
Let  $z_0 = x_0 + iy_0$  and  $w_0 = u_0 + i v_0$ .

Then :

$$(1) \lim_{z \rightarrow z_0} f(z) = \lim_{x+iy \rightarrow x_0+iy_0} f(z) = u_0 + i v_0$$

} complex limit

if and only if



$$(2) \lim_{(x,y) \rightarrow (x_0,y_0)} u(x,y) = u_0 \quad \text{and} \quad \lim_{(x,y) \rightarrow (x_0,y_0)} v(x,y) = v_0.$$

---

$x+iy = 1+i$   
 $(x,y) = (1,1)$

Ex:

$$\lim_{z \rightarrow 1+i} z^2 = \lim_{x+iy \rightarrow 1+i} [(x^2 - y^2) + i 2xy]$$

$$= \lim_{(x,y) \rightarrow (1,1)} [x^2 - y^2] + i \lim_{(x,y) \rightarrow (1,1)} [2xy]$$

$$= [1^2 - 1^2] + i [2(1)(1)] = 2i$$

proof: You can try (1)  $\Rightarrow$  (2).

(pg 8)

(2)  $\Rightarrow$  (1)

Let  $\varepsilon > 0$ .

Suppose  $\lim_{(x,y) \rightarrow (x_0,y_0)} u(x,y) = u_0$  and  $\lim_{(x,y) \rightarrow (x_0,y_0)} v(x,y) = v_0$ .

So there exist  $\delta_1 > 0$  so that if

$$0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta_1 \quad \text{and } (x,y) \in A$$

$$0 < |(x,y) - (x_0,y_0)| < \delta_1$$

then  $|u(x,y) - u_0| < \frac{\varepsilon}{2}$ .

And there exists  $\delta_2 > 0$  so that if

$$0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta_2 \quad \text{and } (x,y) \in A$$

then  $|v(x,y) - v_0| < \frac{\varepsilon}{2}$ .

Note:

$\mathbb{R}^2$

$$\begin{aligned} & \sqrt{(x-x_0)^2 + (y-y_0)^2} \\ &= |(x,y) - (x_0,y_0)| \end{aligned}$$

$\mathbb{C}$

$$\begin{aligned} & \sqrt{(x-x_0)^2 + (y-y_0)^2} \\ &= |z - z_0| \\ & z = x + iy, \quad z_0 = x_0 + iy_0 \end{aligned}$$

So if  $z \in A$  and  $0 < |z - z_0| < \min\{\delta_1, \delta_2\}$  then

pg 9

$$\sqrt{(x-x_0)^2 + (y-y_0)^2}$$

$$|f(x, y) - (u_0 + i v_0)|$$

$$= |u(x, y) + i v(x, y) - u_0 - i v_0|$$

$$= |(u(x, y) - u_0) + i(v(x, y) - v_0)|$$

$$\leq |u(x, y) - u_0| + |i(v(x, y) - v_0)|$$

$$= |u(x, y) - u_0| + \underbrace{|i|}_{1} |v(x, y) - v_0|$$

$$= |u(x, y) - u_0| + |v(x, y) - v_0|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

So,  
 $\lim_{z \rightarrow z_0} f(z) = u_0 + i v_0$

