Proof of ①/②: Assumption: $\lim_{z\to z_0} f(z)=F,\ \lim_{z\to z_0} g(z)=G \Longrightarrow$ ① $\lim_{z\to z_0}[f(z)+g(z)]=F+G$, ② $\lim_{z\to z_0}\alpha f(z)=\alpha F$

Lets prove: If $\lim_{z\to z_0} f(z)=F$ and $\lim_{z\to z_0} g(z)=G$ and $\alpha,\beta\in\mathbb{C}$, then $\lim_{z\to z_0}[\alpha f(z)+\beta g(z)]=\alpha F+\beta G$

proof: Let $\varepsilon>0$.

Note that

$$\left|(\alpha f(z)+\beta g(z))-(\alpha F+\beta G)\right| = \left|(\alpha f(z)-\alpha F)+(\beta g(z)-\beta G)\right| \overset{\triangle}{\le} |\alpha f(z)-\alpha F|+|\beta g(z)-\beta G|$$

$$=|\alpha|\underbrace{|f(z)-F|}+|\beta|\underbrace{|g(z)-G|}_{\text{can control these}} \rightarrow$$

want: $< |\alpha|\cdot\frac{\varepsilon}{2|\alpha|}+|\beta|\cdot\frac{\varepsilon}{2|\beta|}=\varepsilon$

only issue is if $\alpha=0$ or $\beta=0$

Recall $f:A\to\mathbb{C}$
$g:A\to\mathbb{C}$

$$<(|\alpha|+1)|f(z)-F|+(|\beta|+1)|g(z)-G|$$

Since $\lim_{z\to z_0} f(z)=F$ we can find $\delta_1>0$ where if $z\in A$ and $0<|z-z_0|<\delta_1$, then $|f(z)-F|<\frac{\varepsilon}{2(|\alpha|+1)}$

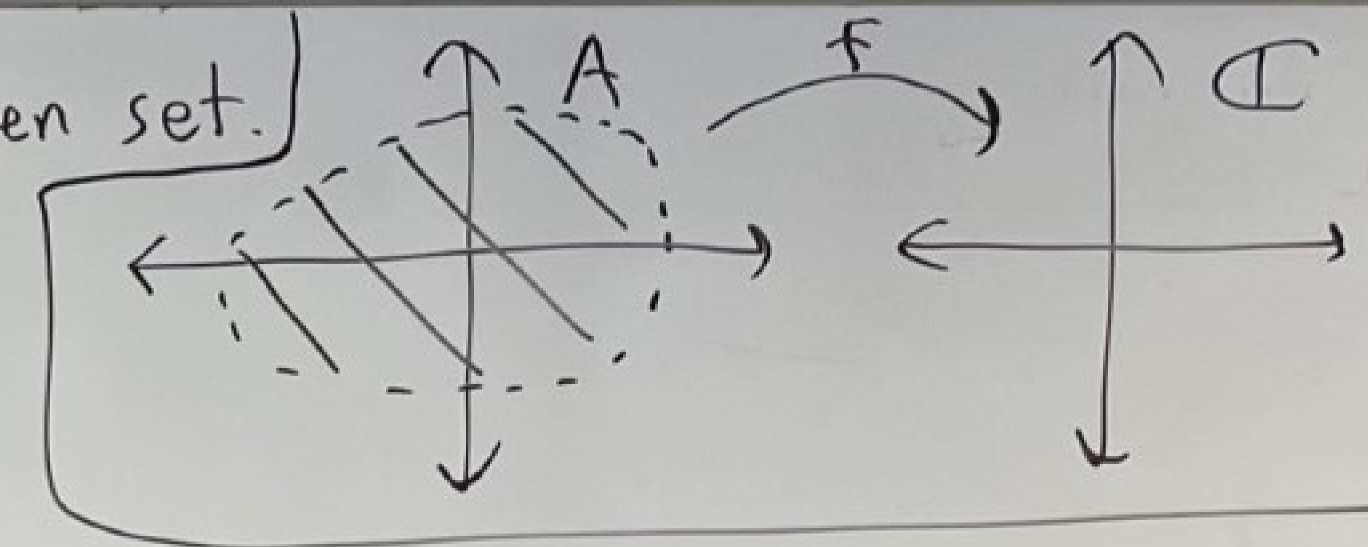
Since $\lim_{z\to z_0} g(z)=G$ we can find $\delta_2>0$ where if $z\in A$ and $0<|z-z_0|<\delta_2$, then $|g(z)-G|<\frac{\varepsilon}{2(|\beta|+1)}$

Let $\delta=\min\{\delta_1,\delta_2\}$. Then if $z\in A$ and $0<|z-z_0|<\delta$ then

$$\left|(\alpha f(z)+\beta g(z))-(\alpha F+\beta G)\right|<(|\alpha|+1)|f(z)-F|+(|\beta|+1)|g(z)-G|<(|\alpha|+1)\frac{\varepsilon}{2(|\alpha|+1)}+(|\beta|+1)\frac{\varepsilon}{2(|\beta|+1)}=\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$

Thus, $\lim(\alpha f(z)+\beta g(z))=\alpha F+\beta G$ $\boxtimes$

Def: Let $A \subseteq \mathbb{C}$ where A is an open set.



Let $f: A \to \mathbb{C}$.

We say that f is <u>continuous</u> at $z_0 \in A$ if $\boxed{\lim_{z \to z_0} f(z) \text{ exists}}$ and $\boxed{\lim_{z \to z_0} f(z) = f(z_0)}$.

We say that f is <u>continuous on A</u> if f is continuous at every $z_0 \in A$

Ex: Let $f(z) = z^2$ and $z_0 = 1+i$. Recall $f(x+iy) = (x+iy)^2 = (x^2-y^2) + i(2xy)$

Then, $\lim_{z \to 1+i} (z^2) = \lim_{x+iy \to 1+i} \left[(x^2-y^2) + i(2xy)\right] = \lim_{(x,y)\to(1,1)} (x^2-y^2) + i \lim_{(x,y)\to(1,1)} (2xy) = (1^2-1^2) + i(2(1)(1))$

$= 2i$

thm from last week

x^2-y^2 and $2xy$ are continuous at $(1,1)$ Calculus III 4670 ?

And, $f(1+i) = (1+i)^2 = 1 + 2i + i^2 = 1 + 2i - 1 = 2i$

So, $\lim_{z \to 1+i} f(z) = f(1+i)$. So, f is cts at $1+i$.

Corollary: (to the thm from last week)

Suppose $f : A \to \mathbb{C}$ and $A \subseteq \mathbb{C}$ is open.

Suppose $f(x+iy) = u(x,y) + i\,v(x,y)$ and $z_0 = x_0 + iy_0$ is in A.

Then f is continuous at z_0 if and only if

$u(x,y)$ is continuous at (x_0,y_0) and $v(x,y)$ is continuous at (x_0,y_0).

$\mathbb{R}^2$ continuity in Calc III or real analysis

proof: Try if you like for practice $\square$

Ex: $f(z) = z^2$

$f(x+iy) = (x^2-y^2) + i(2xy)$ is continuous on all of $\mathbb{C}$ because x^2-y^2 and $2xy$ are continuous on all of $\mathbb{R}^2$.

Theorem: Let $A \subseteq \mathbb{C}$ be an open set and $f : A \to \mathbb{C}$ and $g : A \to \mathbb{C}$. Let $z_0 \in A$. Suppose f and g are continuous at z_0.

Then:

1. $f+g$ is continuous at z_0
2. αf is continuous at z_0 if $\alpha \in \mathbb{C}$.
3. fg is continuous at z_0
4. $\frac{f}{g}$ is continuous if $g(z_0) \neq 0$.

---

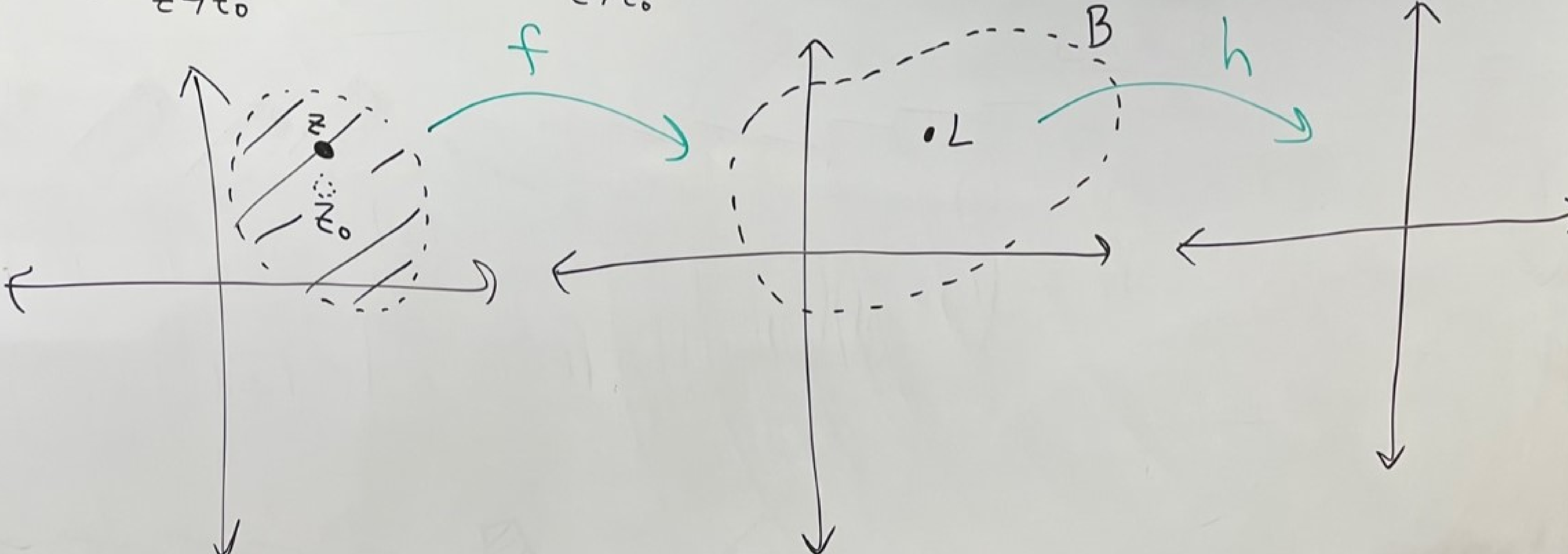
proof: Use theorem from Weds. For example to prove 3 we have

$$\lim_{z \to z_0} (fg)(z) = \lim_{z \to z_0} \left[f(z)g(z)\right] \underset{\boxed{\text{Weds thm}}}{=} \underbrace{\left(\lim_{z \to z_0} f(z)\right)\left(\lim_{z \to z_0} g(z)\right)}_{\text{these exist since } f \text{ and } g \text{ are cts at } z_0} \underset{\boxed{\text{since } f \text{ and } g \text{ are cts at } z_0}}{=} f(z_0)\, g(z_0) = (fg)(z_0)$$

So, fg is cts at z_0 ∎

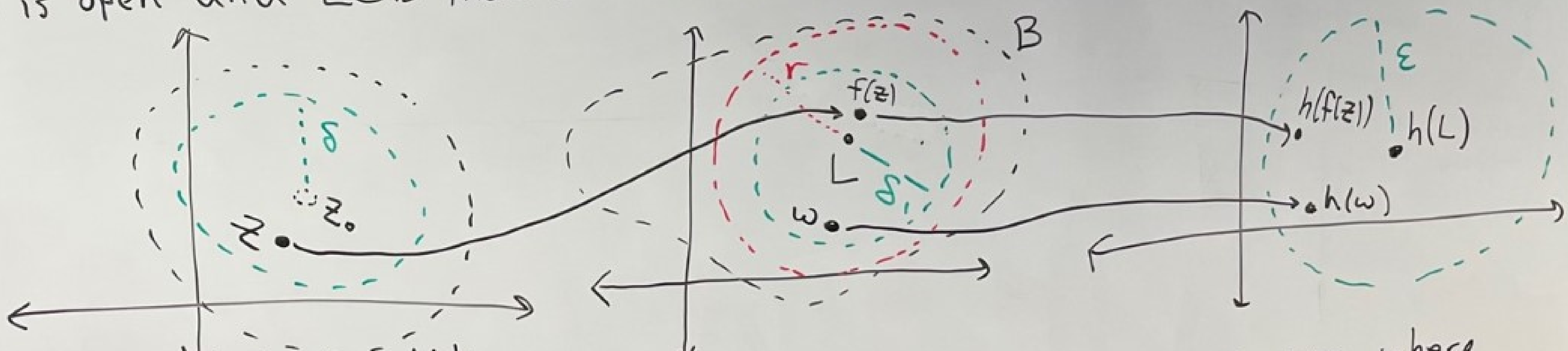
Theorem: Suppose that $\lim_{z \to z_0} f(z) = L$ where f is defined on some deleted neighborhood of z_0. Suppose h is defined on an open set $B \subseteq \mathbb{C}$ where $L \in B$. Suppose h is continuous at L.

Then, $\lim_{z \to z_0} h(f(z)) = h\left(\lim_{z \to z_0} f(z)\right) = h(L)$



proof: Let $\varepsilon > 0$.

Since B is open and $L \in B$ there exists $r > 0$ where $D(L; r) \subseteq B$.



Since h is continuous at L we know $\lim\limits_{w \to L} h(w) = h(L)$, thus there exists $\underbrace{0 < \delta_1 < r}_{\text{will make } w \in B \text{ when } 0<|w-L|<\delta_1}$ where

if $0 < |w - L| < \delta_1$, then $|h(w) - h(L)| < \varepsilon$

Since $\lim\limits_{z \to z_0} f(z) = L$ there exists $\delta > 0$ where if $0 < |z - z_0| < \delta$, then $|f(z) - L| < \delta_1$

[Make δ small enough so $D^*(z_0; \delta)$ is in the domain of f.]

So if $0 < |z - z_0| < \delta$, then $|h(\underbrace{f(z)}_{w}) - h(L)| < \varepsilon$

So, $\lim\limits_{z \to z_0} h(z) = h(L)$ $\square$