Similar to 2(e, f, g)

$$\boxed{a^b = e^{b\log(a)}}$$

$$2\pi - \frac{\pi}{3} = \frac{6\pi - \pi}{3} = \frac{5\pi}{3}$$

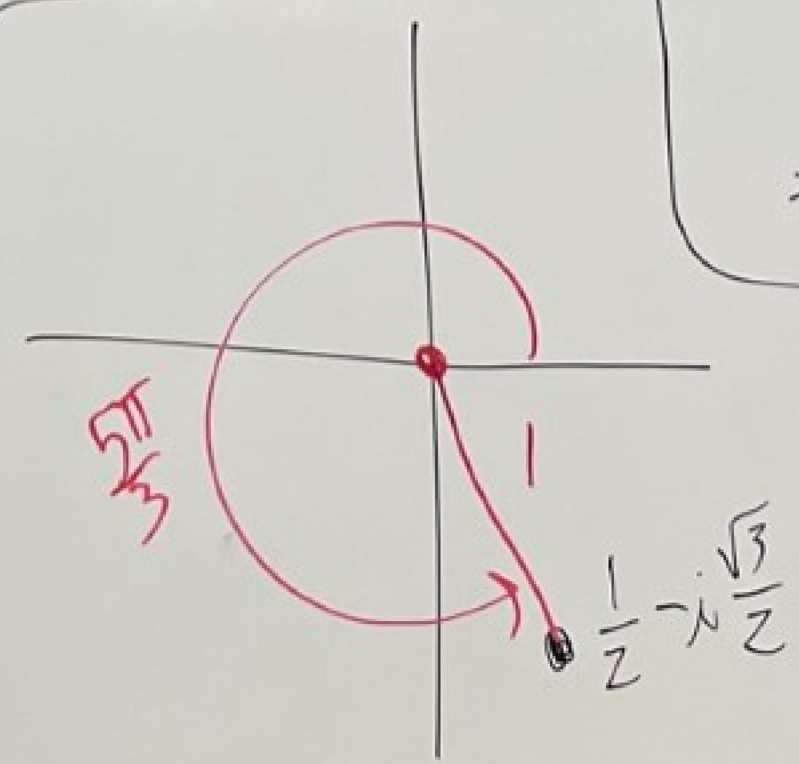
branch of log: $[0, 2\pi)$

Compute

$$\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)^{1+i} = e^{(1+i)\log\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)} = e^{(1+i)\left[\overbrace{\ln\left|\frac{1}{2} - i\frac{\sqrt{3}}{2}\right|}^{\ln(1)=0} + i\frac{5\pi}{3}\right]} = e^{i\frac{5\pi}{3} - \frac{5\pi}{3}}$$

$$= e^{-\frac{5\pi}{3}}\left[\underbrace{\cos\left(\frac{5\pi}{3}\right) + i\sin\left(\frac{5\pi}{3}\right)}_{e^{i\frac{5\pi}{3}}}\right]$$

$$= e^{-\frac{5\pi}{3}}\left[\frac{1}{2} - i\frac{\sqrt{3}}{2}\right] = \boxed{\frac{e^{-\frac{5\pi}{3}}}{2} - i\,\frac{e^{-\frac{5\pi}{3}}\sqrt{3}}{2}}$$



$$\left|\frac{1}{2} - i\frac{\sqrt{3}}{2}\right| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$$

HW 2 Use branch $[-\pi, \pi)$

(12) Prove: Let $z \in \mathbb{C}, z \neq 0$. Show $\log(z) = 0$ iff $z = 1$.

proof:

($\Leftarrow$) Let $z = 1$. Then, $\log(1) = \ln|1| + i0 = 0$

($\Rightarrow$) Suppose $\log(z) = 0$ ← using branch gives 0 in range of log

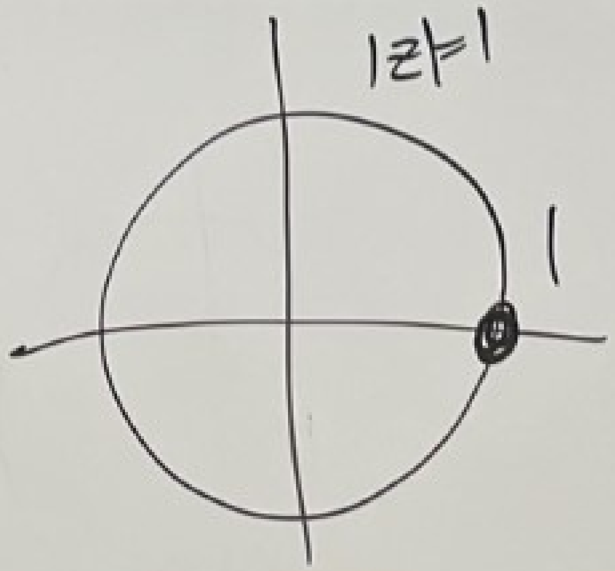
Then $\ln|z| + i\arg(z) = 0$

So, $\ln|z| = 0$ and $\arg(z) = 0$

So, $|z| = 1$ and $\arg(z) = 0$

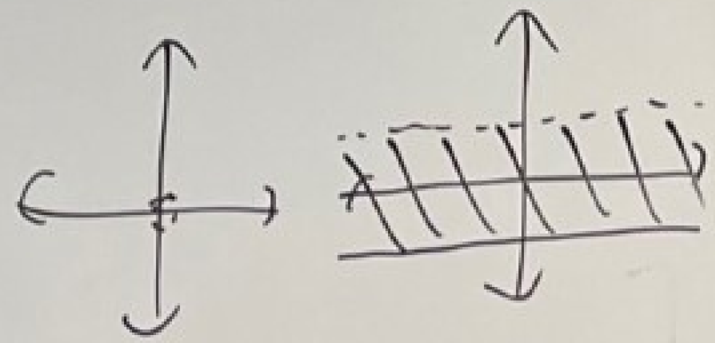
Thus, $z = 1$.

∎



branch: $[2\pi, 4\pi)$

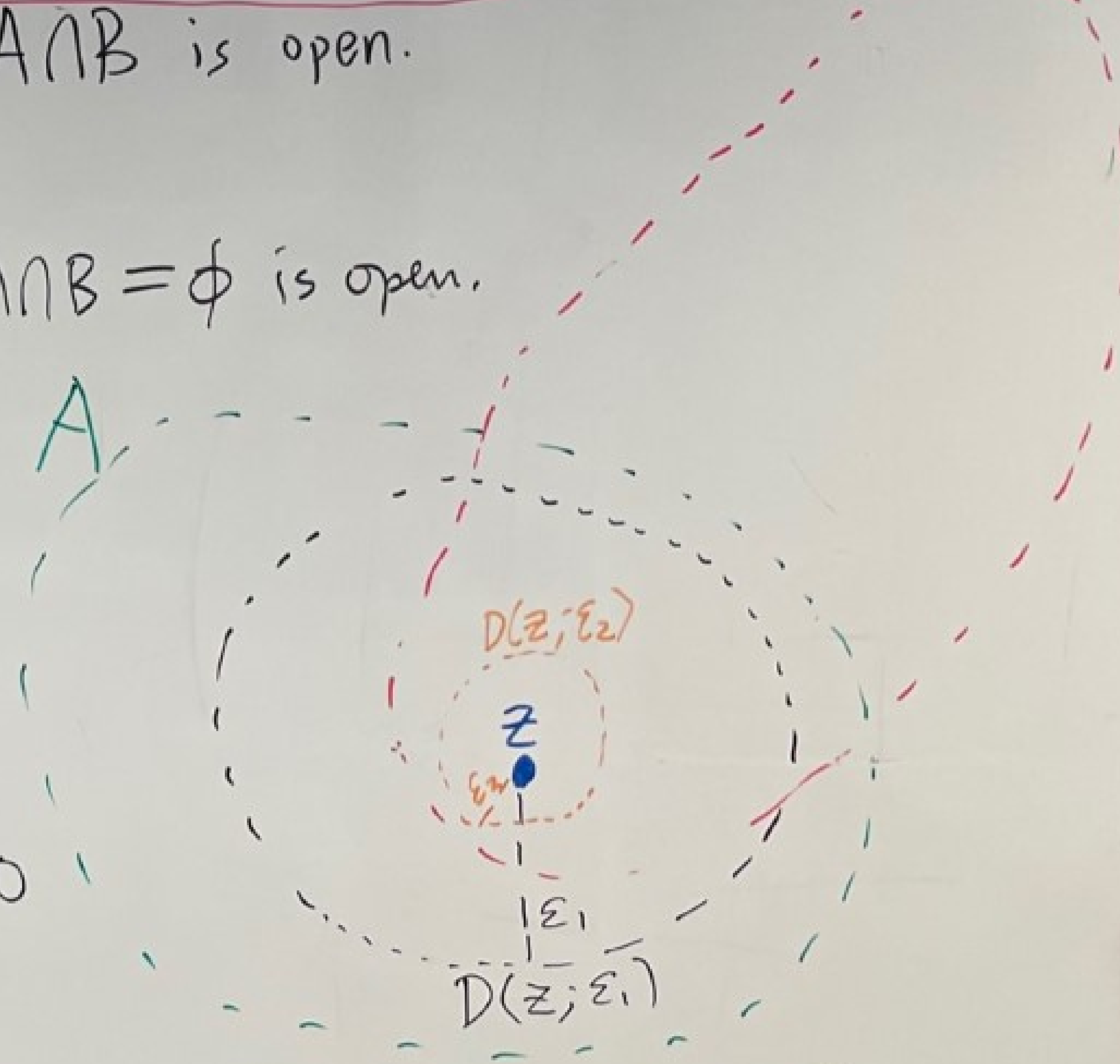
$$\log(1) = \ln|1| + 2\pi i = 2\pi i$$



HW 3 − #3(f)

We used$^{(*)}$: $D(z;r) \subseteq D(z;\varepsilon)$ if $r \le \varepsilon$

Let $A, B \subseteq \mathbb{C}$ be open sets. Then, $A \cap B$ is open.



proof:

If $A \cap B = \emptyset$, then by 3(b) we have $A \cap B = \emptyset$ is open.

So assume $A \cap B \neq \emptyset$.

Let $z \in A \cap B$. We must show z is an interior point of $A \cap B$.

Since $z \in A \cap B$ we know $z \in A$ and $z \in B$.

Since $z \in A$ and A is open there exists $\varepsilon_1 > 0$ where $D(z;\varepsilon_1) \subseteq A$.

Since $z \in B$ and B is open there exists $\varepsilon_2 > 0$ where $D(z;\varepsilon_2) \subseteq B$.

Let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$. Then $\overset{(*)}{D}(z;\varepsilon) \subseteq A \cap B$. So, z is an interior point of $A \cap B$ $\square$

HW 1 #9

Let $z_1, z_2, z_3, z_4 \in \mathbb{C}$ with $|z_3| \neq |z_4|$

Then,

$$\left|\frac{z_1+z_2}{z_3+z_4}\right| \leq \frac{|z_1|+|z_2|}{\big||z_3|-|z_4|\big|}$$

proof: Note $|z_1+z_2| \overset{\Delta}{\leq} |z_1|+|z_2|$ and $|z_3+z_4| \overset{\text{another thm}}{\geq} \big||z_3|-|z_4|\big|$.

This gives $\frac{1}{|z_3+z_4|} \leq \frac{1}{\big||z_3|-|z_4|\big|}$. We know $\big||z_3|-|z_4|\big| \neq 0$ since $|z_3| \neq |z_4|$. Can $|z_3+z_4| = 0$?

If $|z_3+z_4| = 0$ then $z_3+z_4 = 0$ and then $z_3 = -z_4$ and then $|z_3| = \underbrace{|-z_4|}_{|-1||z_4| = |z_4|}$ and then $|z_3| = |z_4|$. But $|z_3| \neq |z_4|$.

So, the eqn above is ok.

Thus, $$\left|\frac{z_1+z_2}{z_3+z_4}\right| = \frac{|z_1+z_2|}{|z_3+z_4|} \leq \frac{|z_1|+|z_2|}{|z_3+z_4|} \leq \frac{|z_1|+|z_2|}{\big||z_3|-|z_4|\big|}$$ ∎

HW 2 #5

What does $f(z)=\frac{1}{z}$ map $S=\{z \mid |z|<1\}$ to?

Suppose $|z|<1$.

Then, $\left|\frac{1}{z}\right| = \frac{|1|}{|z|} = \frac{1}{|z|} > 1$.

Let $T=\{w \mid |w|>1\}$.

We just showed $f(S) \subseteq T$.

Let's show $f(S)=T$.

Let $w \in T$. Then, $|w|>1$.

Then, $\frac{1}{|w|}<1$. So, $\left|\frac{1}{w}\right|<1$.

Thus, $\frac{1}{w} \in S$ and $f\left(\frac{1}{w}\right) = \frac{1}{\left(\frac{1}{w}\right)} = w$.

So, $f(S)=T$. $\square$