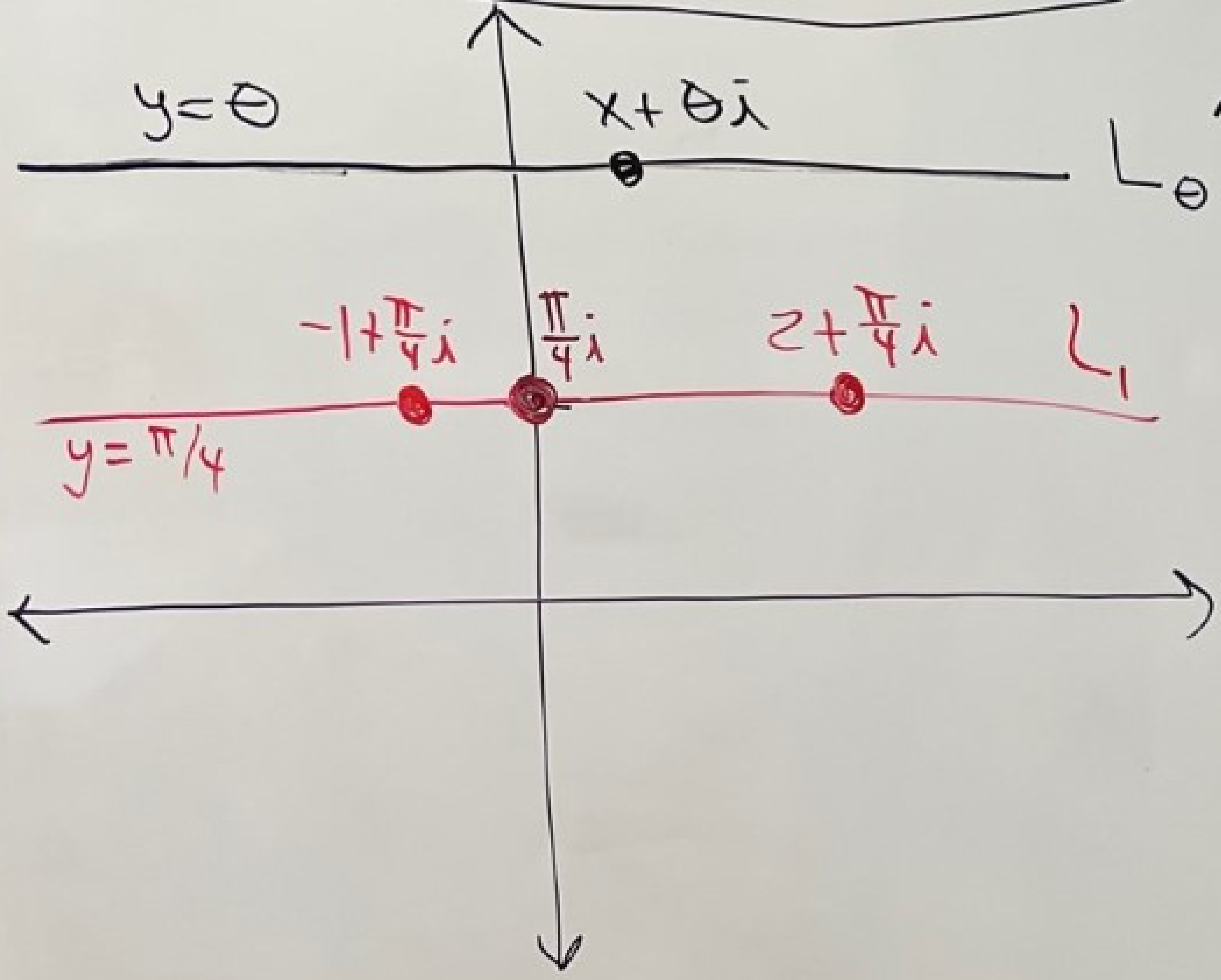
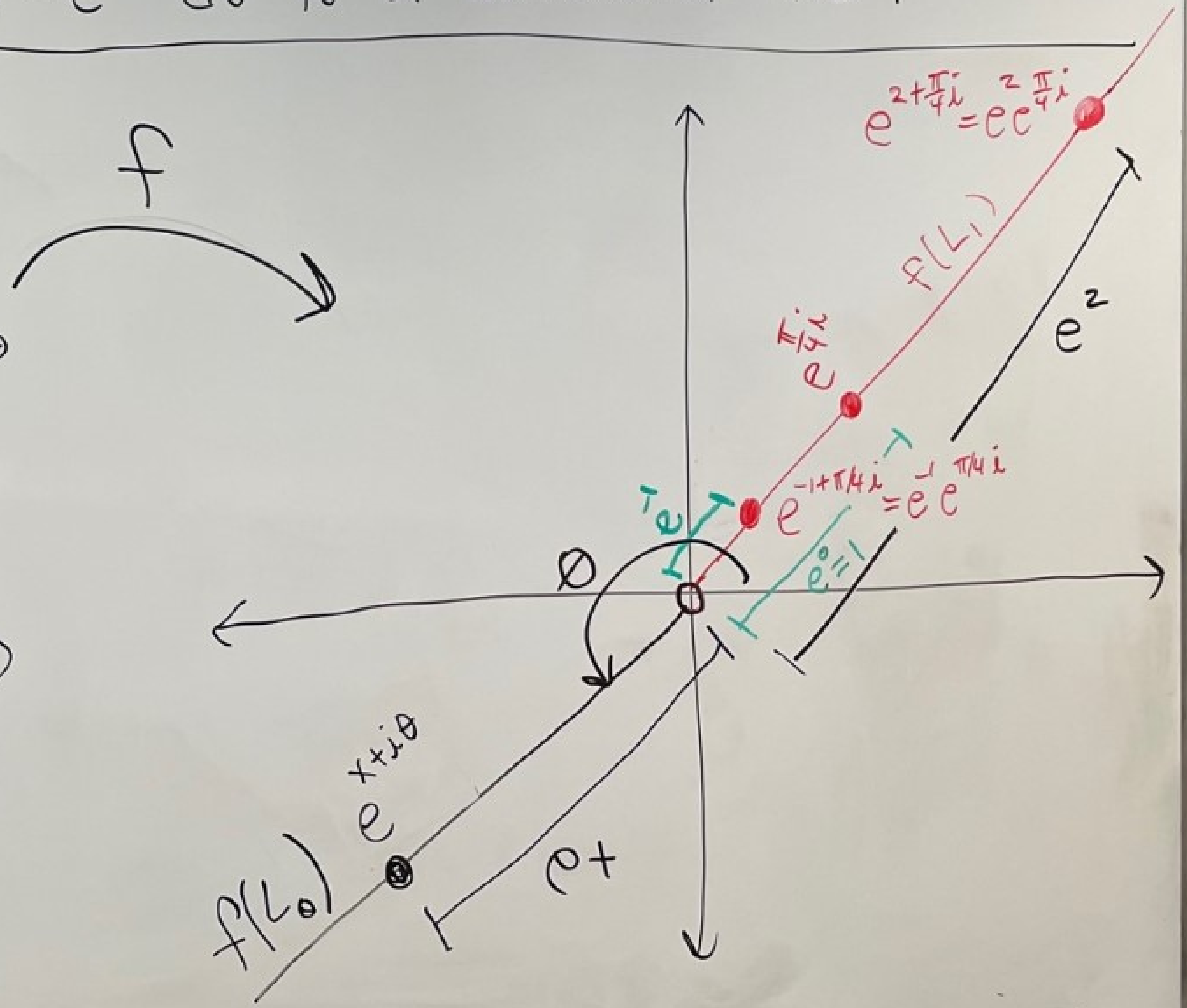
Ex: What does $f(z)=e^z$ do to a horizontal line?

$$e^{x+iy} = \underbrace{e^x}_{\text{distance from } 0}\Big[\underbrace{\cos(y)+i\sin(y)}_{y \text{ is the angle}}\Big]$$

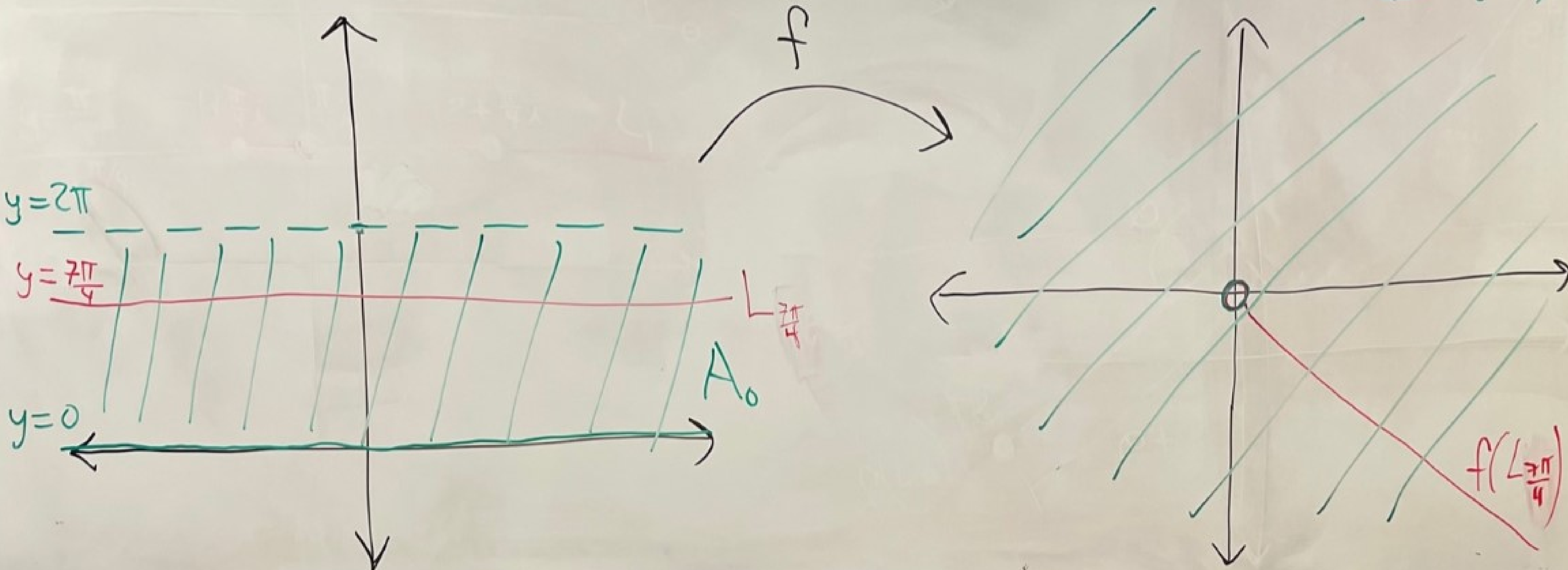




Thus, $f(z) = e^z$ maps

$$A_0 = \{ x + iy \mid x, y \in \mathbb{R},\ 0 \leq y < 2\pi \}$$

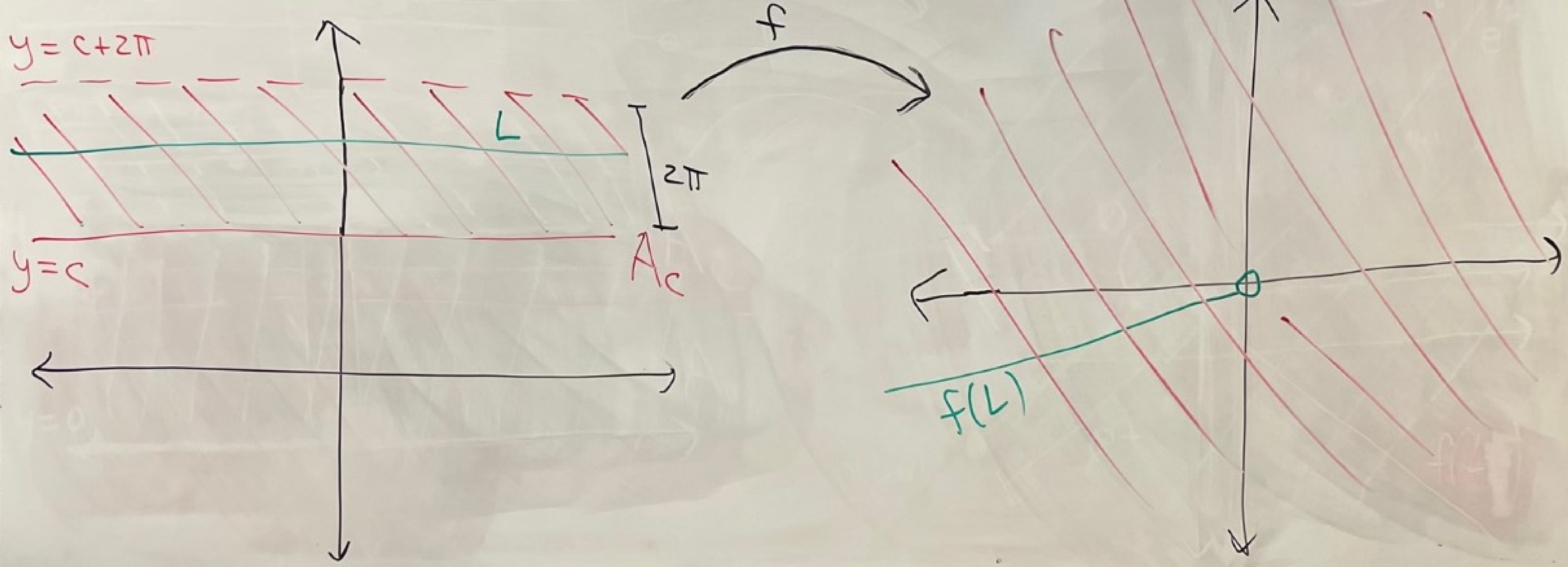
onto $\mathbb{C} - \{0\}$ in a 1-1 and onto way



Thus, $f(z) = e^z$ maps

$$A_c = \{ x + iy \mid x, y \in \mathbb{R},\ c \le y < c + 2\pi$$

onto $\mathbb{C} - \{0\}$ in a 1-1 and onto way



Square function

Let $f : \mathbb{C} \to \mathbb{C}$ be $f(z) = z^2$

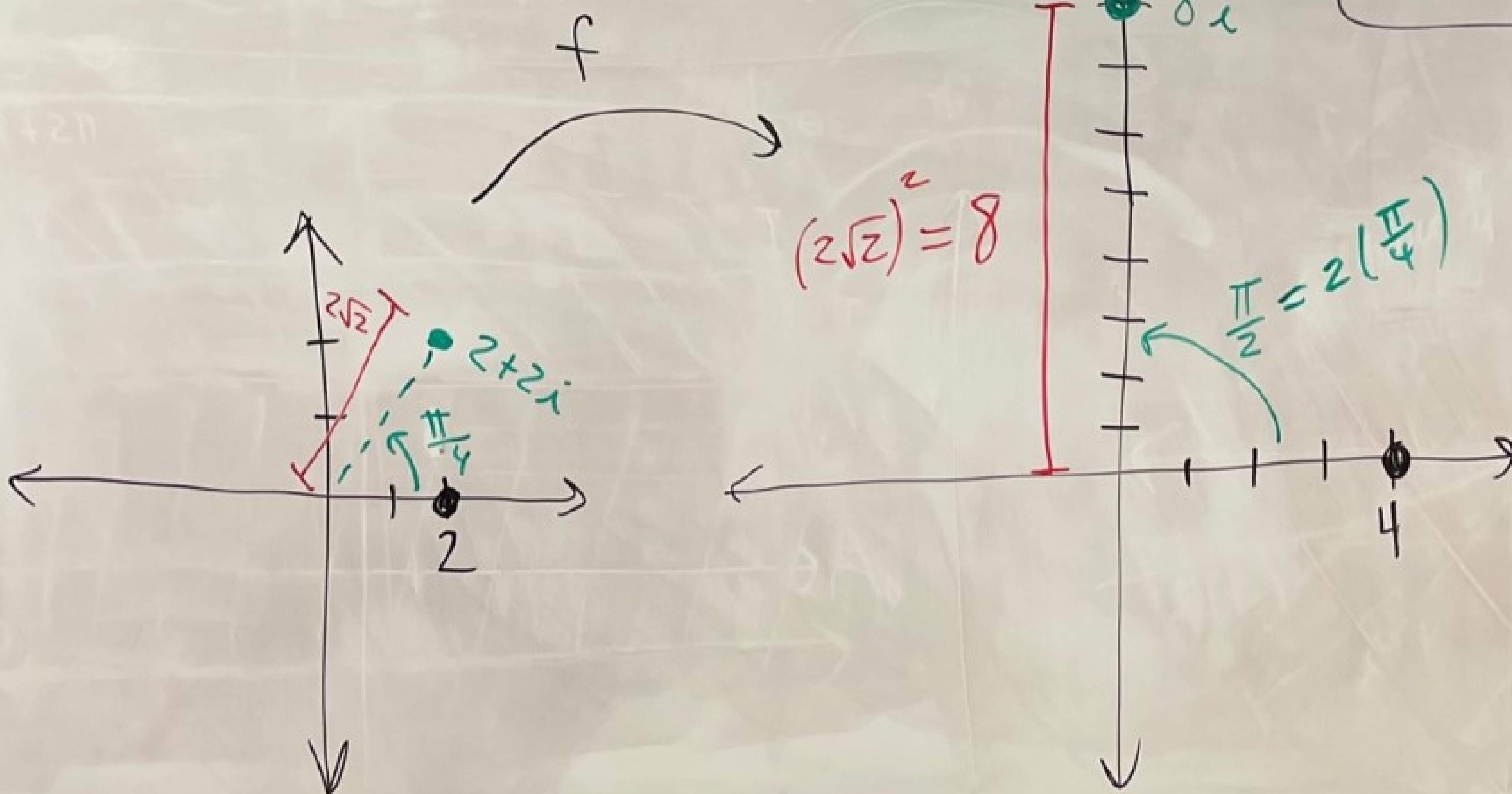
$f(2) = 2^2 = 4$

$f(2+2i) = (2+2i)(2+2i)$

$= 4 + 4i + 4i + 4i^2$

$= 8i$

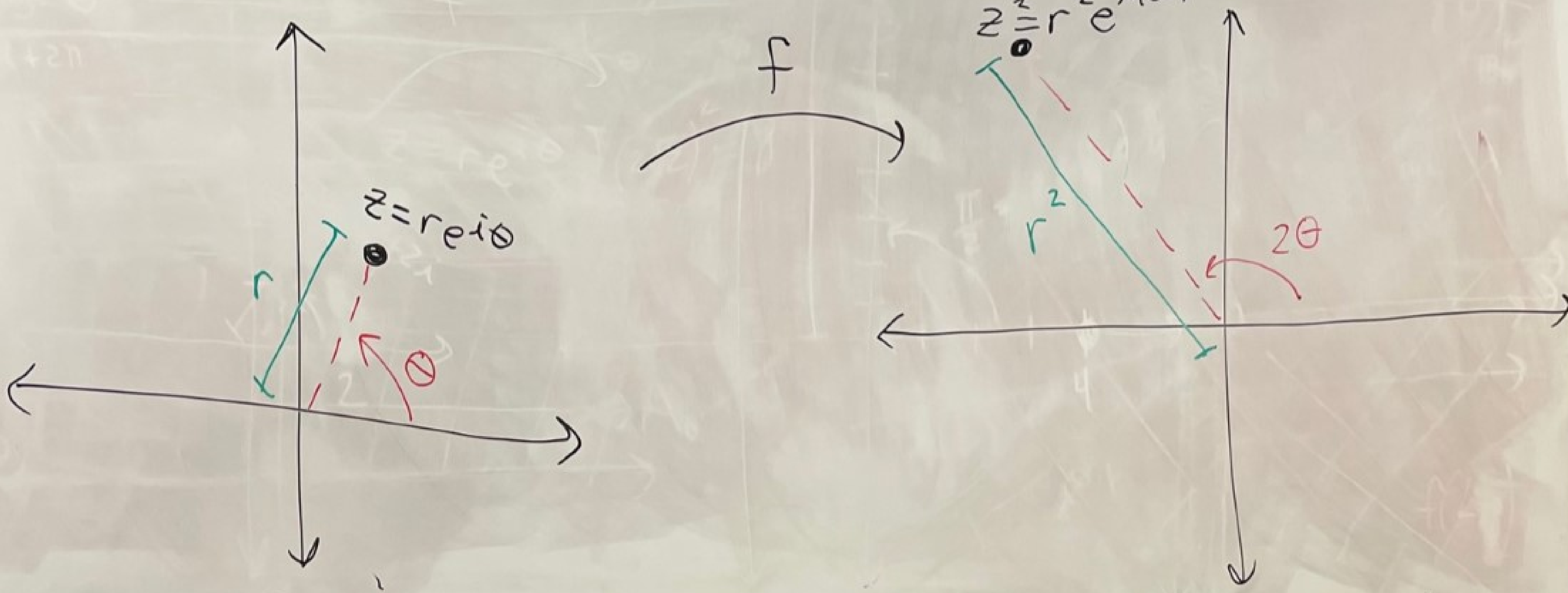
$i^2 = -1$



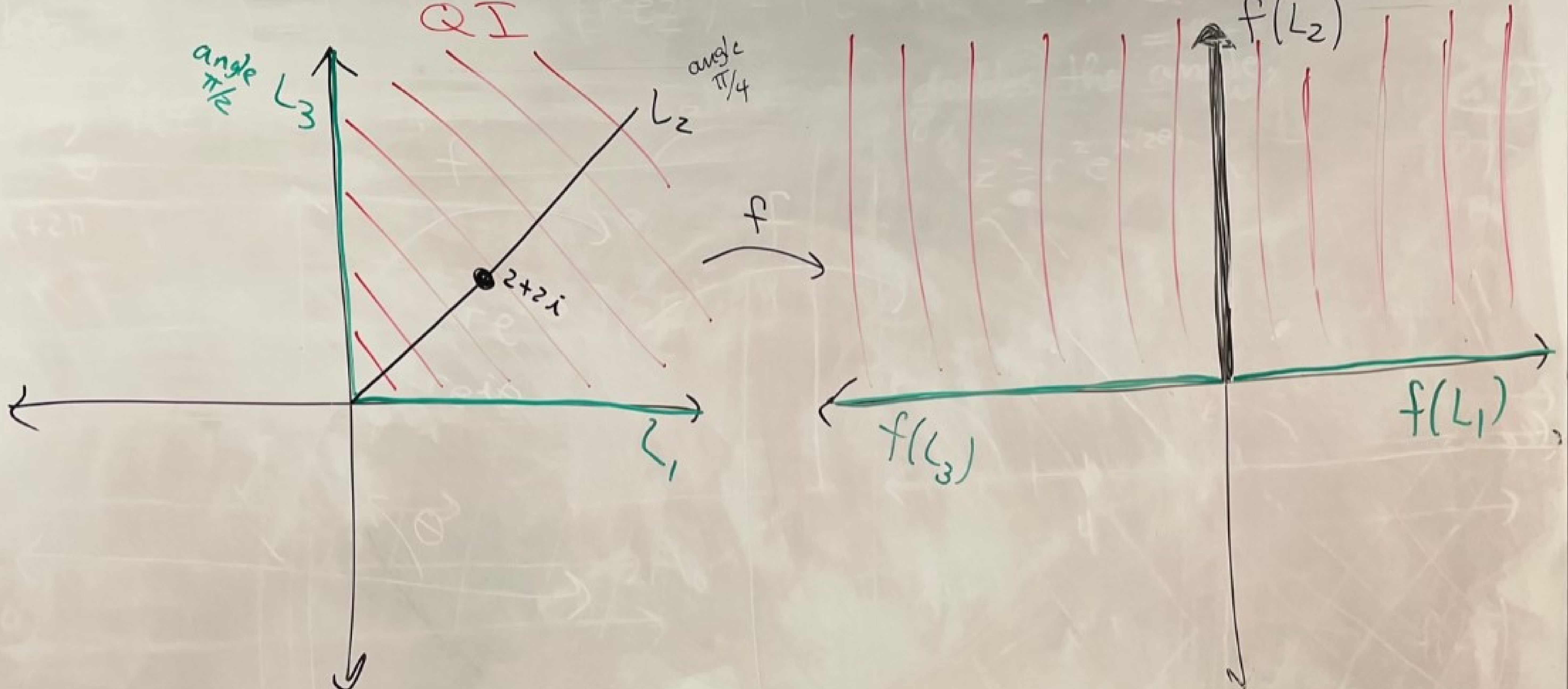
Let $z = re^{i\theta}$.

Then, $f(z) = f(re^{i\theta}) = (re^{i\theta})^2 = re^{i\theta} re^{i\theta} = r^2 e^{i\theta + i\theta} = \boxed{r^2 e^{i(2\theta)}}$

So, $f(z) = z^2$ squares the distance and doubles the angle.



What does $f(z) = z^2$ do to the 1st quadrant?



Trig functions

We have $\cos(\theta)$ and $\sin(\theta)$ when $\theta \in \mathbb{R}$.

Let's extend to the complex plane.

Let $\theta \in \mathbb{R}$.

Then,

$$\boxed{e^{i\theta} = \cos(\theta) + i\sin(\theta).} \quad (1)$$

and

$$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta)$$

so,

$$\boxed{e^{-i\theta} = \cos(\theta) - i\sin(\theta)} \quad (2)$$

Computing (1) + (2) gives

$$e^{i\theta} + e^{-i\theta} = 2\cos(\theta)$$

So,

$$\boxed{\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}}$$

Computing (1) − (2) gives

$$e^{i\theta} - e^{-i\theta} = 2i\sin(\theta)$$

So,

$$\boxed{\sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}}$$

Def: Let $z \in \mathbb{C}$.

Define

$$\cos(z) = \frac{e^{iz}+e^{-iz}}{2}$$

and

$$\sin(z) = \frac{e^{iz}-e^{-iz}}{2i}$$

This definition extends $\cos(\theta)$ and $\sin(\theta)$ to all of $\mathbb{C}$. Ie it agrees with real-valued sine and cosine when z is real.

Ex:

$$\cos(\pi+i) = \frac{e^{i(\pi+i)}+e^{-i(\pi+i)}}{2} \overset{i^2=-1}{=} \frac{e^{i\pi-1}+e^{-i\pi+1}}{2}$$

$$= \frac{e^{-1+\pi i}+e^{1-\pi i}}{2}$$

$$= \frac{e^{-1}\left[\overbrace{\cos(\pi)}^{-1}+i\overbrace{\sin(\pi)}^{0}\right]+e^{1}\left[\overbrace{\cos(-\pi)}^{-1}+i\overbrace{\sin(-\pi)}^{0}\right]}{2}$$

$$= \frac{-e^{-1}-e}{2} = \left(\frac{-\frac{1}{e}-e}{2}\right) = -\left(\frac{\frac{1}{e}+e}{2}\right)$$

Theorem: Let $z, w \in \mathbb{C}$.

Then:

1. $\sin(-z) = -\sin(z)$
2. $\cos(-z) = \cos(z)$
3. $\sin^2(z) + \cos^2(z) = 1$
4. $\sin(z+w) = \sin(z)\cos(w) + \cos(z)\sin(w)$
5. $\cos(z+w) = \cos(z)\cos(w) - \sin(z)\sin(w)$