Math 4740
10/2/24

Recall: E and F are independent if
$P(E \cap F) = P(E)P(F)$

Ex: Suppose you roll two 6-sided dice, one red and one green.
Let E be the event that the green die is 1.
Let F be the event that the red die is 6.
Are these events independent?

$E = \{(1,1), (2,1), (3,1), (4,1), (5,1), (6,1)\}$

$F = \{(6,1), (6,2), (6,3), (6,4), (6,5), (6,6)\}$

$E \cap F = \{(6,1)\}$

$|S| = 36$

$$P(E \cap F) = \frac{|E \cap F|}{|S|} = \frac{1}{36}$$

$$P(E) \cdot P(F) = \frac{|E|}{|S|} \cdot \frac{|F|}{|S|} = \frac{6}{36} \cdot \frac{6}{36} = \frac{1}{36}$$

So, $P(E \cap F) = P(E) \cdot P(F)$

So, E and F are independent.

<u>Ex:</u> Suppose you roll two 6-sided die, one red and one green.

Let E be the event that the sum of the dice is 6.
Let F be the event that the red die is 4.
Are these events independent.

$E = \{(1,5),(2,4),(3,3),(4,2),(5,1)\}$

$F = \{(4,1),(4,2),(4,3),(4,4),(4,5),(4,6)\}$

$E \cap F = \{(4,2)\}$

$P(E \cap F) = 1/36$

not equal

$P(E) \cdot P(F) = \frac{5}{36} \cdot \frac{6}{36} = \frac{5}{6} \cdot \frac{1}{36}$

So, E and F are <u>not</u> independent.

Theorem Let S be a sample space of a repeatable experiment. Let A and B be two events where $A \cap B = \phi$ [there's no overlap] Suppose further that each time we repeat the experiment S, the experiment is independent of the previous time we did the experiment S. Suppose we keep repeating S until either A or B occurs and then we stop. The probability that A occurs before B is given by

$$\frac{P(A)}{P(A)+P(B)}$$

these are calculated in S

[Derivation in notes online]

Ex: Suppose we continually roll two 6-sided die until the sum of the die is either 6 or 7. What's the probability that the sum will be 6 before the sum will be 7?

Example of Ex:

roll 1: $\boxed{6}$ $\boxed{4}$ sum = 10

roll 2: $\boxed{4}$ $\boxed{1}$ sum = 5

roll 3: $\boxed{2}$ $\boxed{3}$ sum = 5

roll 4: $\boxed{4}$ $\boxed{4}$ sum = 8

roll 5: $\boxed{1}$ $\boxed{5}$ sum = 6 ← stop

Here sum of 6 occured before sum of 7

S is rolling two 6-sided die

$|S| = 36$

$A = \{(1,5),(2,4),(3,3),(4,2),(5,1)\}$ ← Sum is 6

$B = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$ ← Sum is 7

If you keep doing S over and over until A or B occurs, the probability of A before B is

$$\frac{P(A)}{P(A)+P(B)} = \frac{5/36}{5/36 + 6/36} = \boxed{\frac{5}{11}}$$

prob. sum 6 occurs before sum 7

What's the probability that B occurs before A?

$$\frac{P(B)}{P(B)+P(A)} = \frac{6/36}{6/36+5/36} = \boxed{\frac{6}{11}}$$

prob. sum 7 occurs before sum 6

Topic 4 - Random Variables, Expected Value, Games

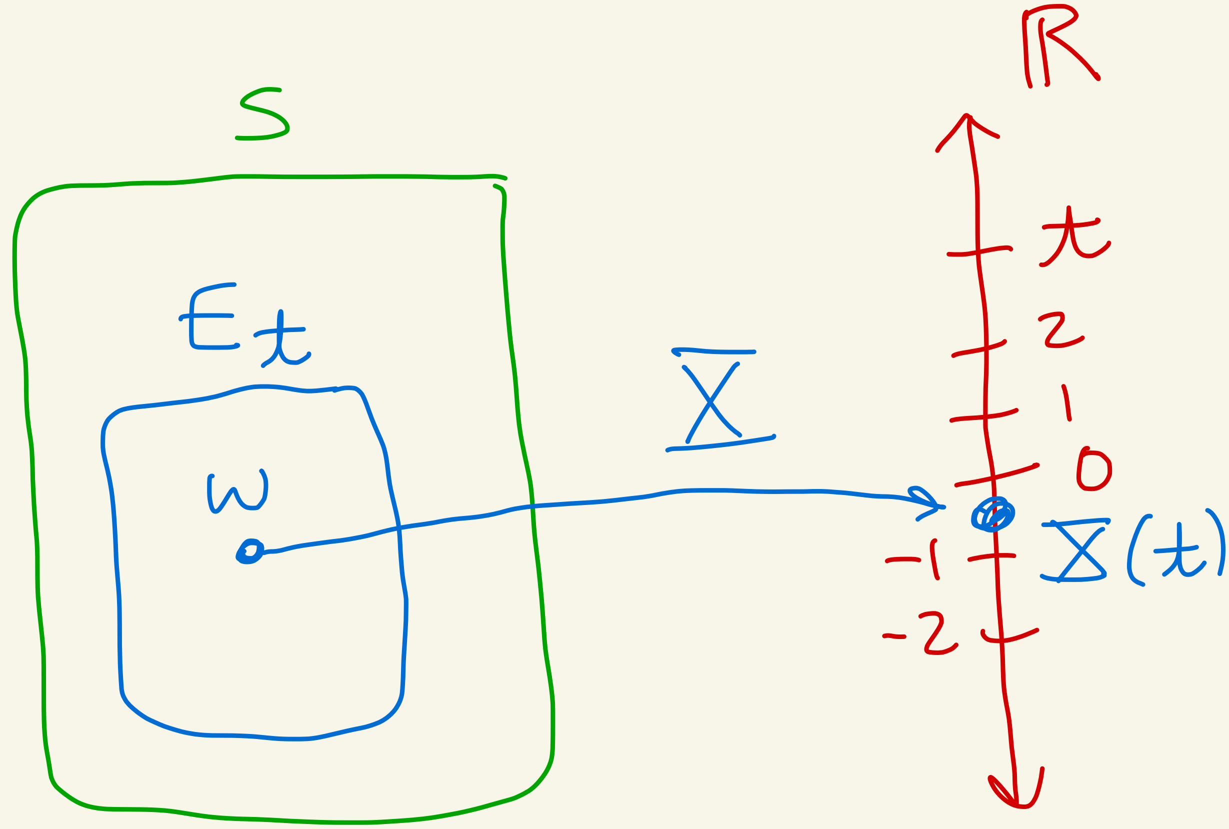
<u>Def:</u> Let (S, Ω, P) be a probability space. A <u>random variable</u> is a function

$$\underbrace{X : S \to \mathbb{R}}_{\text{X is a function from S to $\mathbb{R}$}}$$

such that for all real numbers t we have that

$$E_t = \{\omega \mid \omega \in S \text{ and } X(\omega) \le t\}$$

is an event in Ω



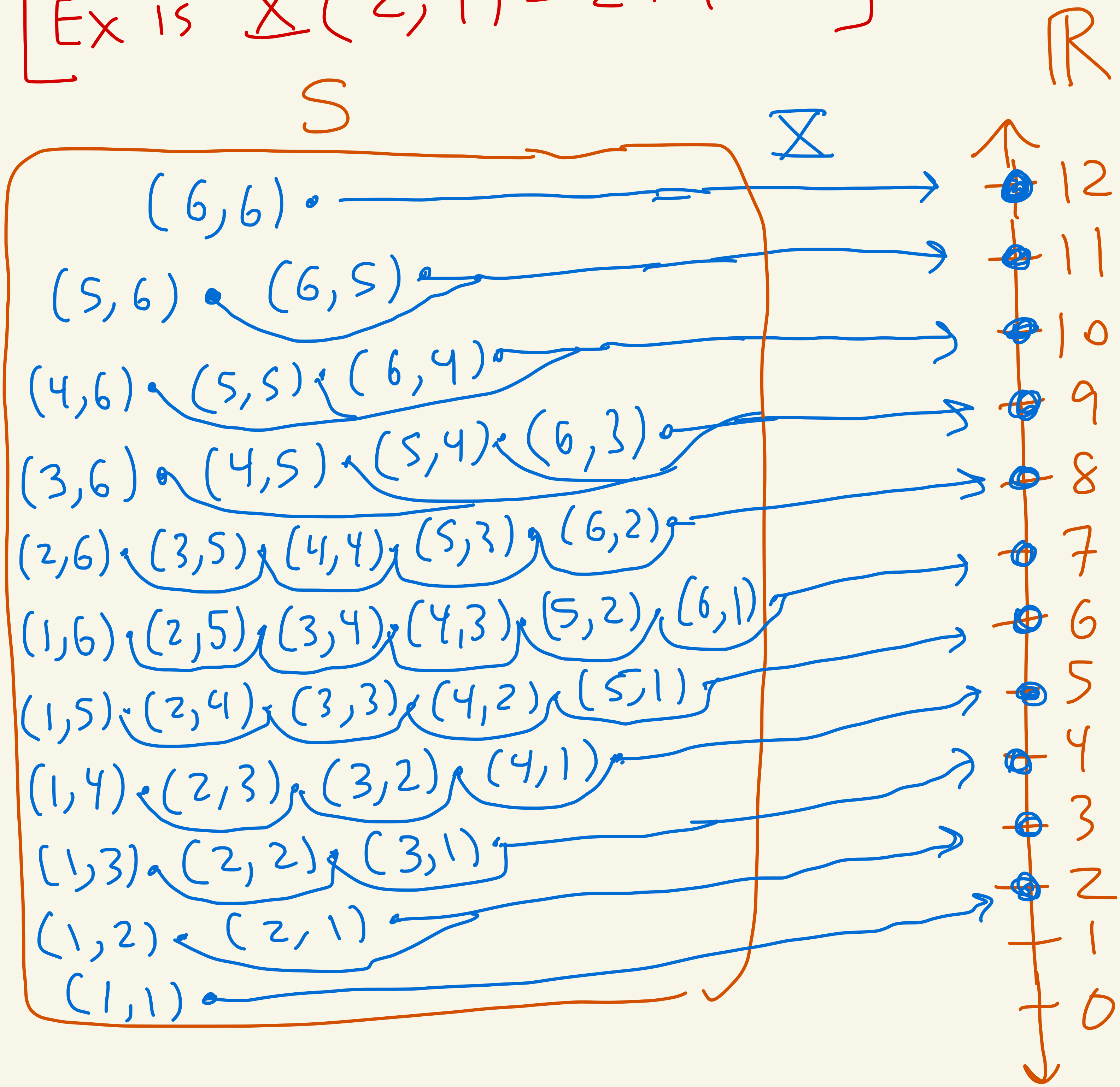
<u>Note:</u> The condition on E_t means we can calculate $P(E_t)$.
In our class when S is finite and Ω is all subsets of S this condition will always be satisfied. So for us, a random variable is just a function

$$X : S \to \mathbb{R}$$

Def: Let $X : S \to \mathbb{R}$ be a random variable. We say that X is *discrete* if the range of X can be enumerated as a list of values: $\underbrace{x_1, x_2, x_3, \ldots}_{\text{can be finite list or infinite list}}$

Ex: Let (S, Ω, P) be a probability space representing rolling two 6-sided die. Let $X: S \to \mathbb{R}$ be the sum of the die.

[Ex is $X(2,4) = 2+4 = 6$]



X is discrete since its range is 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12

<u>Def</u>: Let X be a random variable on a probability space (S, Ω, P).

Define

$$P(X = i) = \underbrace{P(\{\omega \mid \omega \in S \text{ where } X(\omega) = i\})}_{\text{probability of all } \omega \text{ where } X(\omega) = i}$$

$$P(X \leq i) = \underbrace{P(\{\omega \mid \omega \in S \text{ and } X(\omega) \leq i\})}_{\text{probability of all } \omega \text{ where } X(\omega) \leq i}$$

Similarly you can define

$P(X < i)$, or $P(X \geq i)$, etc.

The probability function p of X is $p(i) = P(X = i)$

The cumulative distribution function of X is

$$F(i) = P(X \leq i)$$

So, $F : \mathbb{R} \to \mathbb{R}$.