Math 4740
10/23/24

Topic 5 – Binomial Random Variables

A Bernoulli trial is an experiment with two possible outcomes: success or failure.

Suppose success occurs with probability p. Then failure would have probability $1-p$.

Ex: Experiment is flipping a coin.

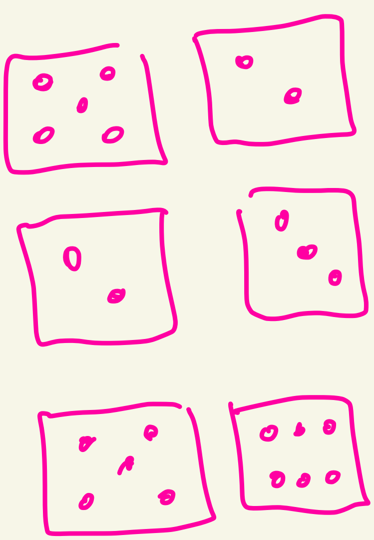
success = tails, $p = \frac{1}{2}$

failure = heads, $1-p = \frac{1}{2}$

Ex: Rolling two 6-sided dice.

Let success be that the sum of the dice is 7 or 11.

And failure is any other sum



← sum is 7, success

← sum is 5, failure

← sum is 11, success

The probability of success is

$$p = \underbrace{6/36}_{\substack{\text{probability}\\ \text{of sum}\\ \text{being 7}}} + \underbrace{2/36}_{\substack{\text{probability}\\ \text{of sum}\\ \text{being 11}}} = 8/36$$

The probability of failure is

$$1 - p = 1 - 8/36 = \frac{28}{36}$$

Ex: Experiment is playing a round of Roulette with the American wheel.

Let success be that the ball lands on black.

Then failure is not black, that is green or red.

$$\overset{\text{success}}{p} = {}^{18}/_{38} = {}^{9}/_{19}$$

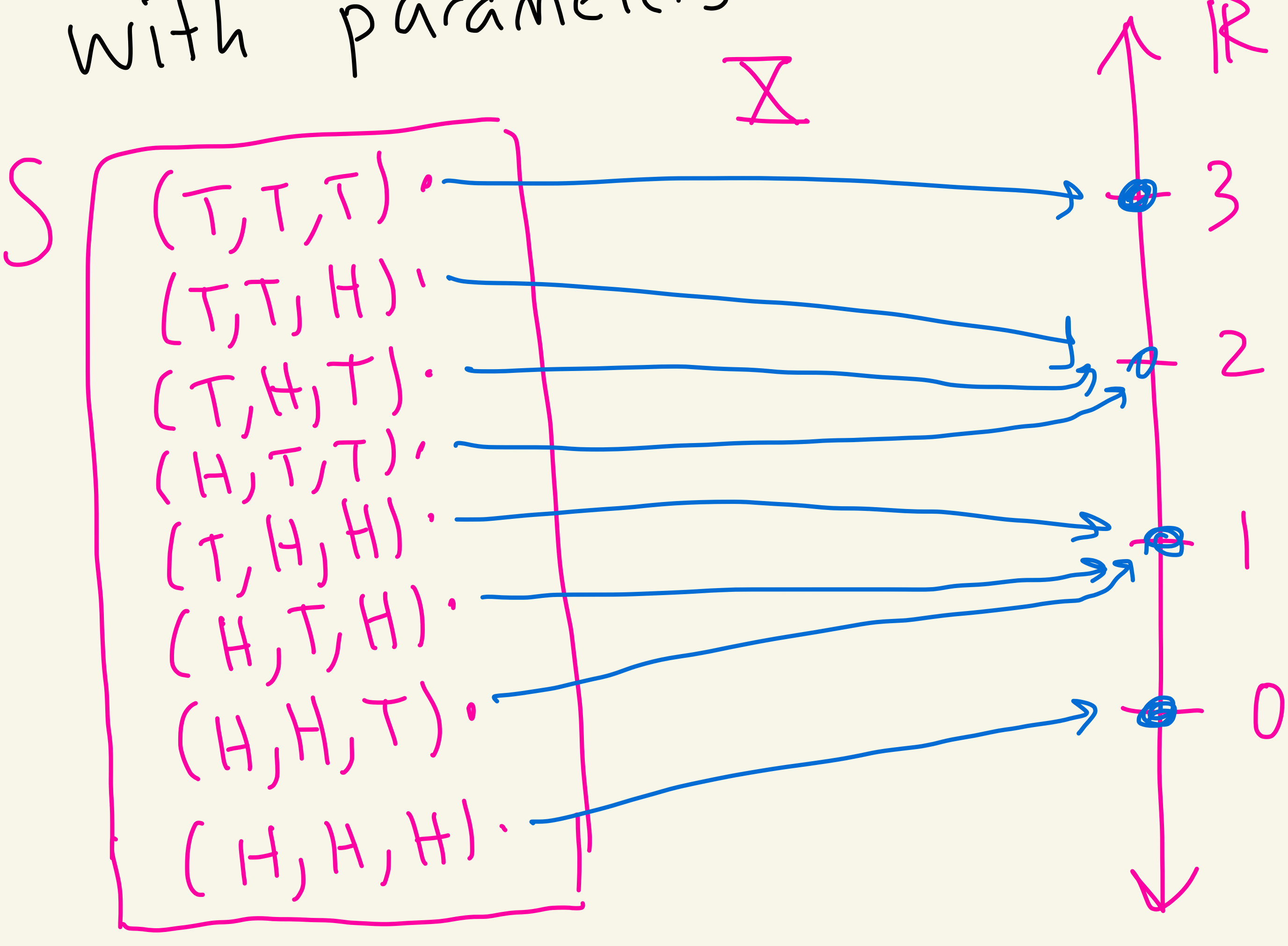
$$\underbrace{1-p}_{\text{failure}} = 1-p = {}^{10}/_{19}$$

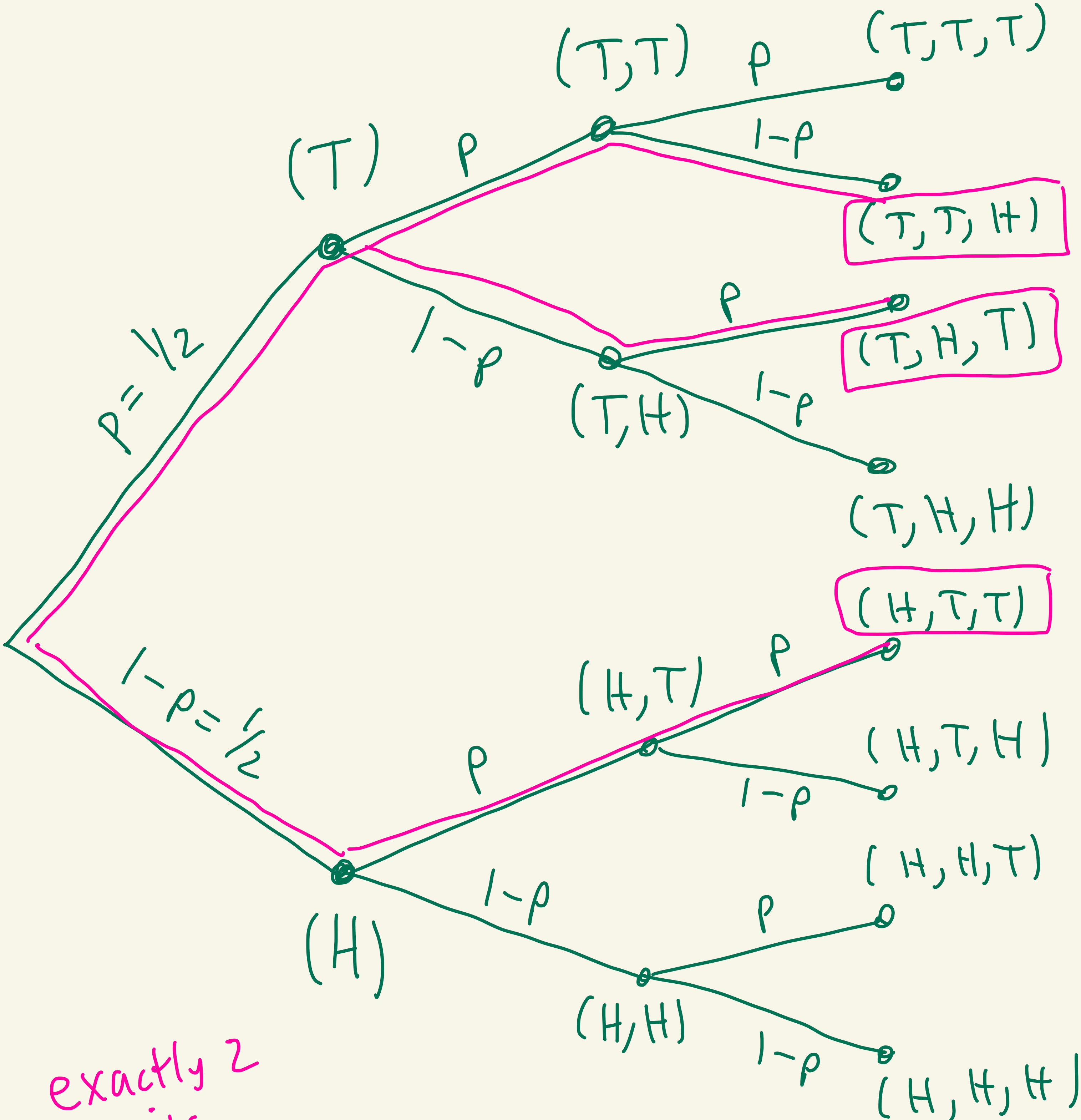
38	total
18	red
18	black
2	green

Now suppose that n Bernoulli trials, each with success p, are performed in a row independently of each other. Let X be the number of successes. Then X is a <u>binomial random variable with parameters n and p.</u>

Ex: Suppose you flip a coin $n = 3$ times. Where on each flip success is T (tails) with probability $p = 1/2$.

Let X be the total number of tails. Then X is a binomial random variable with parameters $n = 3$, $p = \frac{1}{2}$.





$$P(\overbrace{X=2}^{\text{exactly 2 tails}}) = p \cdot p \cdot (1-p) + p(1-p)p$$

$$+ (1-p)pp$$

$$= 3p^2(1-p)^1$$

$$= \underbrace{\binom{3}{2}}_{\text{pick 2 spots for the tails out of 3}} \underbrace{p^2}_{\text{2 successes}} \underbrace{(1-p)^1}_{\text{1 failure}}$$

We can generalize the above as follows:

Theorem: Let X be a binomial random variable with parameters n and p. Then,

$$\underbrace{P(X=k)}_{\text{prob. of exactly k successes}} = \binom{n}{k} p^k (1-p)^{n-k}$$

where $0 \leq k \leq n$

Ex: Suppose we flip a coin 100 times. What's the probability of exactly 48 tails occuring?

$n = 100$ flips

success $= T$

failure $= H$

$k = \#$ successes $= 48$

$p = \frac{1}{2} \leftarrow$ probability of success

$1-p = \frac{1}{2} \leftarrow$ probability of failure

$X =$ total # of tails

$$P(X=48) = \underbrace{\binom{100}{48}}_{\binom{n}{k}} \cdot \underbrace{\left(\frac{1}{2}\right)^{48}}_{p^k} \cdot \underbrace{\left(\frac{1}{2}\right)^{52}}_{(1-p)^{n-k}}$$

$$= \binom{100}{48} \cdot \left(\frac{1}{2}\right)^{100}$$

$$= \frac{93{,}206{,}558{,}875{,}049{,}876{,}949{,}581{,}681{,}100}{1{,}267{,}650{,}600{,}228{,}229{,}401{,}496{,}703{,}205{,}376}$$

$$\approx 0.073527\ldots \approx \boxed{7.35\%}$$

<u>Ex</u>: Suppose we flip a coin 20 times. What is the probability of getting between 10 and 12 tails, that is, 10, 11, or 12 tails?

$n = 20 \leftarrow$ # of flips

$p = \frac{1}{2} \leftarrow$ probability of success (tail)

$1 - p = \frac{1}{2} \leftarrow$ probability of failure (head)

X = total # of tails in the 20 flips

Want

$$P(10 \le X \le 12) =$$

$$= P(X=10) + P(X=11) + P(X=12)$$

$$= \binom{20}{10} \cdot \underbrace{\left(\frac{1}{2}\right)^{10}}_{\text{success}} \underbrace{\left(\frac{1}{2}\right)^{10}}_{\text{failure}} + \binom{20}{11} \cdot \underbrace{\left(\frac{1}{2}\right)^{11}}_{\text{success}} \cdot \underbrace{\left(\frac{1}{2}\right)^{9}}_{\text{failure}}$$

$$+ \binom{20}{12} \cdot \underbrace{\left(\frac{1}{2}\right)^{12}}_{\text{success}} \cdot \underbrace{\left(\frac{1}{2}\right)^{8}}_{\text{failure}}$$

$$= \frac{\binom{20}{10} + \binom{20}{11} + \binom{20}{12}}{2^{20}}$$

$$= \frac{184{,}756 + 167{,}960 + 125{,}970}{1{,}048{,}576}$$

$$\approx 0.456511\ldots \approx \boxed{45.6511\,\%}$$