Math 4740
2/3/25

(Topic 1 continued...)

Ex: Suppose you roll a 6-sided die with sides labeled 1, 2, 3, 4, 5, 6. Through experimentation you realize the sides aren't equally likely. You estimate the following probabilities.

outcome	probability
1	1/4
2	1/8
3	1/8
4	1/16

Note:

$$\frac{1}{4}+\frac{1}{8}+\frac{1}{8}+\frac{1}{16}+\frac{1}{16}+\frac{3}{8}$$

5	1/16
6	3/8

$$= \frac{2}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{3}{8} = 1$$

Let's make a probability space.

$S = \{1, 2, 3, 4, 5, 6\}$ ← all possible outcomes

$\Omega = \{\text{all subsets of } S\}$ ← set of events

$= \{\phi, \{1\}, \{2\}, \ldots, \{1, 4, 5\}, \ldots\}$

Probability function

$p : \Omega \to \mathbb{R}$

Define

$P(\{1\}) = 1/4$ $\quad$ $P(\{4\}) = 1/16$

$P(\{2\}) = 1/8$ $\quad$ $P(\{5\}) = 1/16$

$P(\{3\}) = 1/8$ $\quad$ $P(\{6\}) = 3/8$

Then extend P to all of Ω by disjoint summation, like this:

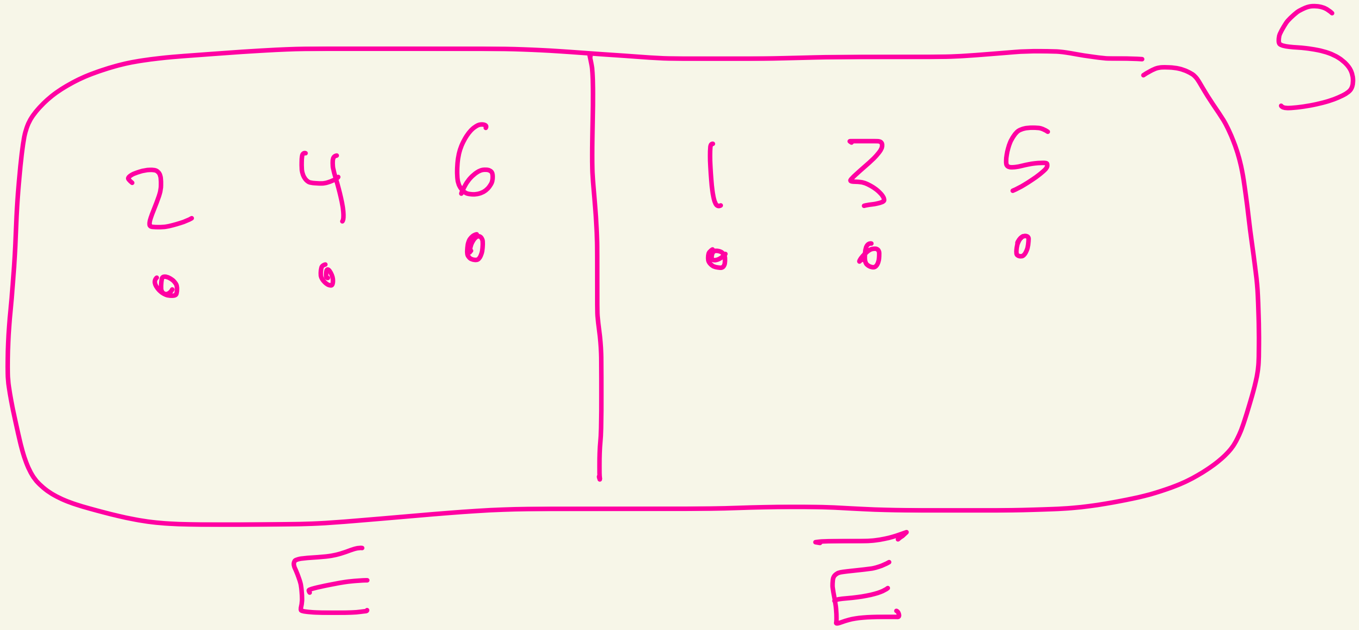
$$\underbrace{P(\{2,4,6\})}_{\text{probability of rolling an even \#}} = P(\{2\}) + P(\{4\}) + P(\{6\})$$
$$= 1/8 + 1/16 + 3/8$$
$$= 9/16 \approx 0.5625$$

What about the probability of getting an odd number?

$$P(\{1,3,5\}) = P(\{1\}) + P(\{3\}) + P(\{5\})$$
$$= 1/4 + 1/8 + 1/16$$

$= \frac{7}{16} = 0.4375$

We got this:



We saw $P(E) = 9/16$

and $P(\bar{E}) = 7/16$.

$P(\bar{E}) = 1 - P(E)$

Note: Suppose (S, Ω, P) is a probability space and S is finite. And suppose each outcome is equally likely.

That is,

$$P(\{w\}) = \frac{1}{|S|}$$

where w is any element of S.

Let $E = \{w_1, w_2, \ldots, w_n\}$ is an event from Ω.

Then

$$P(E) = P(\{w_1\}) + P(\{w_2\}) + \cdots + P(\{w_n\})$$

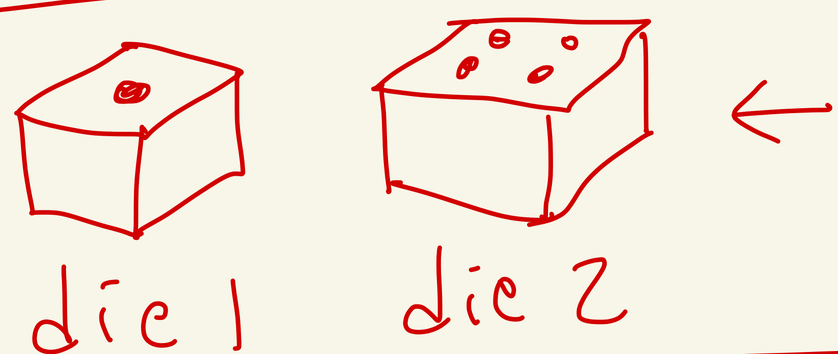
$$= \underbrace{\frac{1}{|S|} + \frac{1}{|S|} + \cdots + \frac{1}{|S|}}_{}$$

n terms

$$= \frac{n}{|S|} = \frac{|E|}{|S|}$$

So, $P(E) = \frac{|E|}{|S|}$ if each outcome of S is equally likely.

<u>Ex:</u> Let's model the experiment of rolling two 6-sided die. They are normal dice, each side is equally likely.



$S = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6),$
$(2,1), (2,2), (2,3), (2,4), (2,5), (2,6),$
$(3,1), (3,2), (3,3), (3,4), (3,5), (3,6),$
$(4,1), (4,2), (4,3), (4,4), (4,5), (4,6),$

$(5,1), (5,2), (5,3), (5,4), (5,5), (5,6),$
$(6,1), (6,2), (6,3), (6,4), (6,5), (6,6)\}$

$\Omega = \{\text{all subsets of } S\}$

$P : \Omega \to \mathbb{R}$

$P(\{(a,b)\}) = \frac{1}{|S|} = \frac{1}{36}$

for any (a,b)

For example, $P(\{(4,4)\}) = \frac{1}{36}$

What is the probability that the sum of the dice is 7?

Let E be the event the sum of the dice is 7.

Then
$E = \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}$

$$P(E) = \frac{|E|}{|S|} = \frac{6}{36}$$

END EX.

<u>Theorem:</u> Let (S, Ω, P) be a probability space. Let E and F be events from Ω.

Then:

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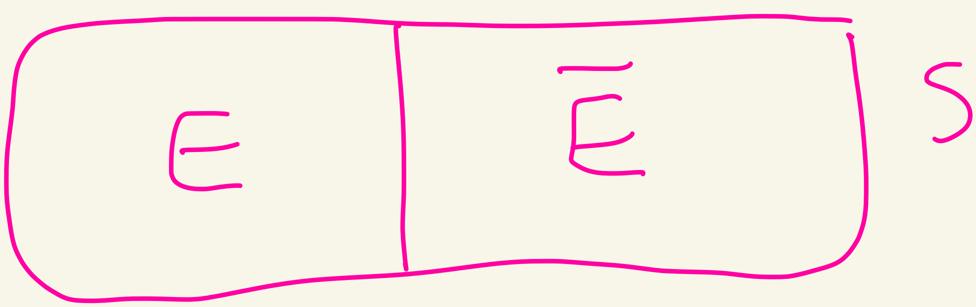
① $P(E) = 1 - P(\bar{E})$



$P(E) + P(\bar{E}) = 1$

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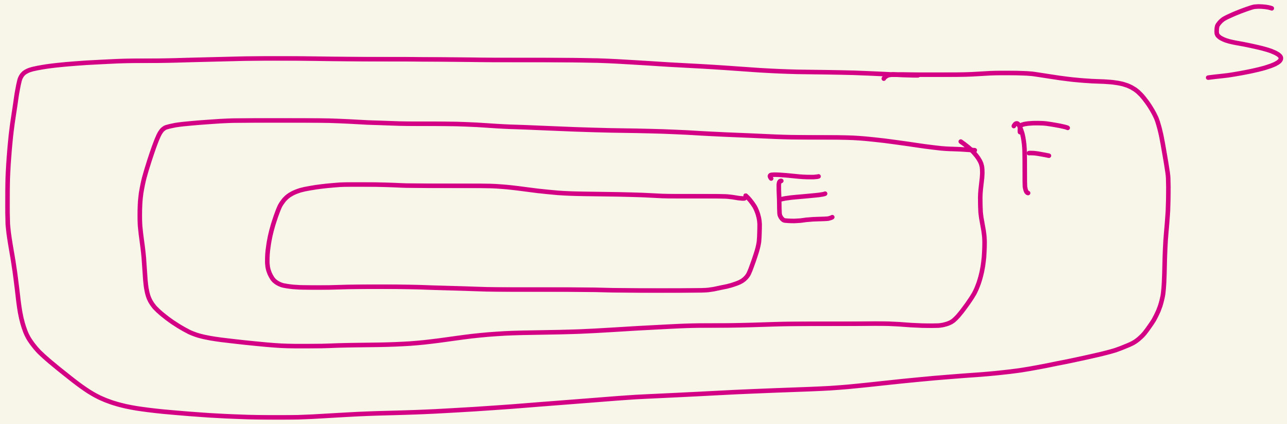
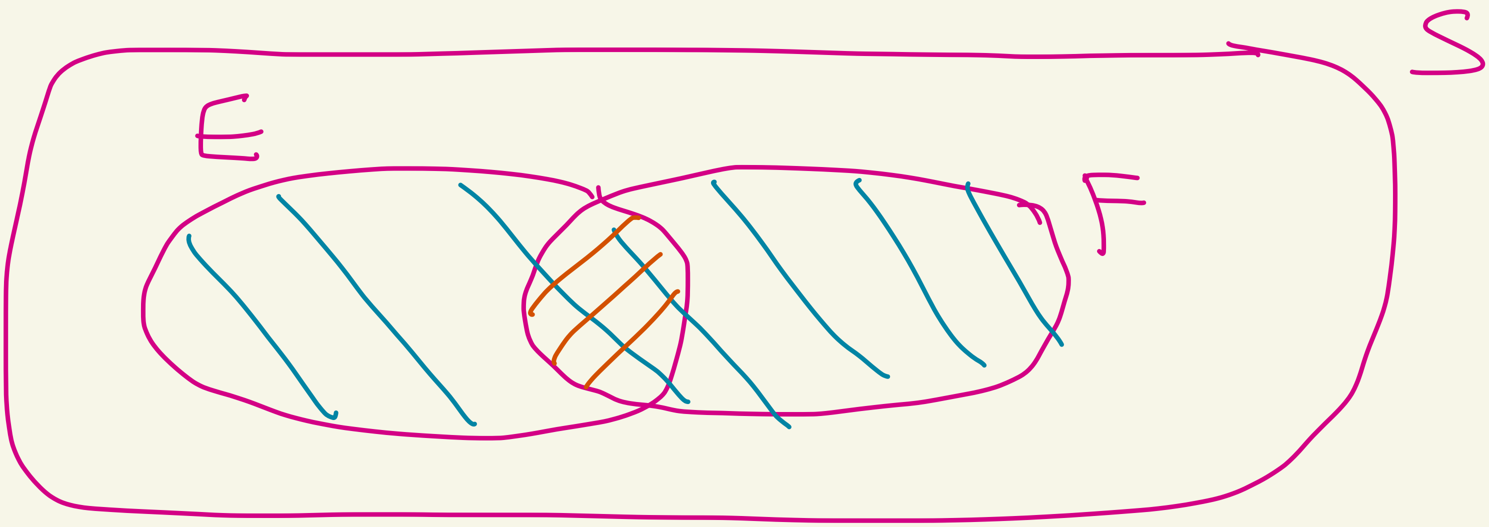
② If $E \subseteq F$, then $P(E) \leq P(F)$.

subset



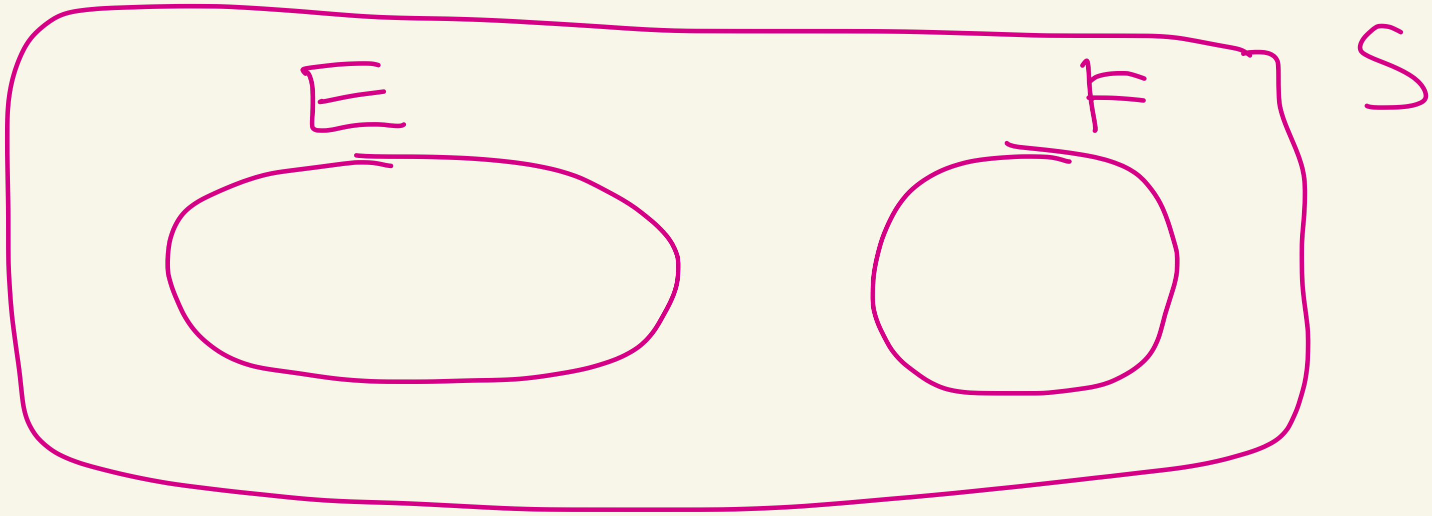
③ $P(E \cup F) = P(E) + P(F) - P(E \cap F)$



\\\\\\ $= E \cup F$ /// $= E \cap F$

④ If $E \cap F = \phi$, then

$P(E \cup F) = P(E) + P(F)$



proof: See online notes

END THEOREM

Ex: Suppose we roll two 12-sided dice. The sides are labeled 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12. Each side is equally likely.

Question: What's the probability that at least one of the dice is greater than or equal to 4?

die 1	die 2	
1	3	✗
4	2	✓ die 1 ≥ 4
10	7	✓ both die ≥ 4

$$S = \left\{ (a,b) \;\middle|\; \begin{array}{l} 1 \le a \le 12 \\ 1 \le b \le 12 \end{array} \right\}$$

$$= \{(1,1), \ldots, (7,11), \ldots, (12,12)\}$$

$$|S| = 12^2 = 144$$

All outcomes equally likely.

Let E be the event that at least one die is greater than or equal to 4.

$$E = \{(1,4), (1,5), (1,6), (1,7), \ldots\}$$

theres too many to list.

Let's instead calculate $\overline{E}$ which is both dice are less than 4.

$$\overline{E} = \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3)\}$$

So, $P(\overline{E}) = \dfrac{|\overline{E}|}{|S|} = \dfrac{9}{144}$

Thus,

$$P(E) = 1 - P(\overline{E}) = 1 - \frac{9}{144} = \frac{135}{144}$$

$$= 0.9375$$

$$= \boxed{93.75\%}$$