Math 4740
8/28/24

Def: Suppose $A_1, A_2, A_3, \dots$ are an infinite number of sets.

Then,

$$\bigcap_{i=1}^{\infty} A_i = A_1 \cap A_2 \cap A_3 \cap \cdots$$

$$= \left\{ x \;\middle|\; \begin{array}{l} x \text{ is in every} \\ \text{one of the } A_i \end{array} \right\}$$

$$\bigcup_{i=1}^{\infty} A_i = A_1 \cup A_2 \cup A_3 \cup \cdots$$

$$= \left\{ x \;\middle|\; \begin{array}{l} x \text{ is in at least} \\ \text{one of the } A_i \end{array} \right\}$$

Def: Let A and B be sets. A **function** f from A to B, notated $f: A \to B$, is a rule that assigns to each element of A a unique element of B

Ex: Let

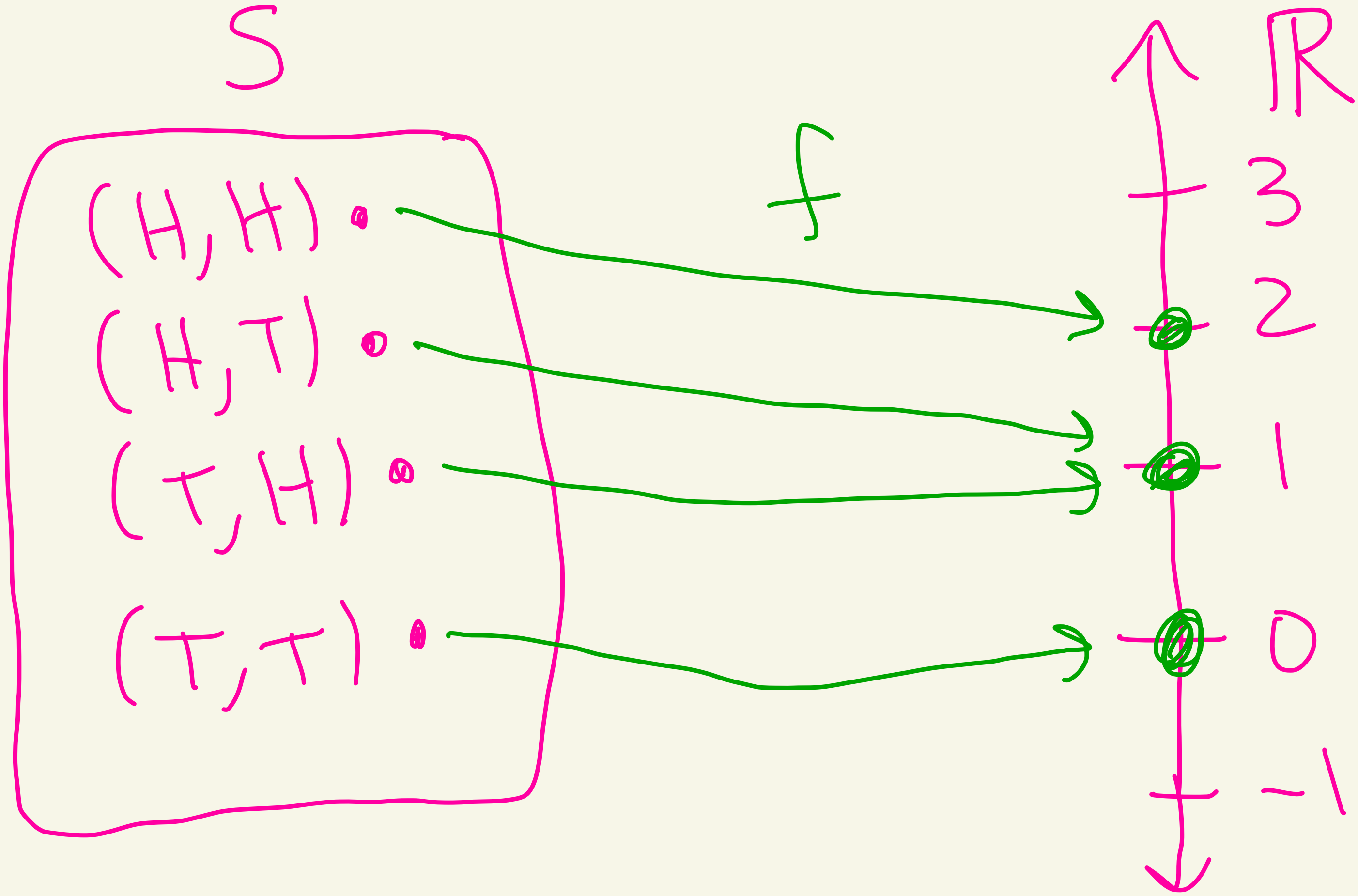
$$S = \{(H,H),(H,T),(T,H),(T,T)\}$$

represent flipping a coin twice.

Let $f: S \to \mathbb{R}$ where f counts how many heads occured.

So,

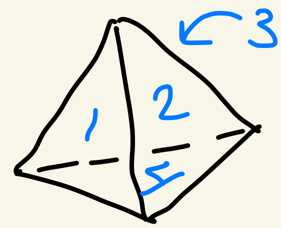
$$f(H,H) = 2 \qquad f(T,H) = 1$$
$$f(H,T) = 1 \qquad f(T,T) = 0$$



Later in the class, f will be called a random variable.

Example of making a probability Space

Suppose we want to model the experiment of rolling a 4-sided die



Let

$$S = \{1, 2, 3, 4\}$$

S is called the sample space of all possible outcomes.

Let

Omega

$$\Omega = \{ \phi, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}, \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}, \{1,2,3,4\} \}$$

Ω is the set that contains all the subsets of S.

Ω is called the set of events.
Ω contains all the events that we want to be able to measure the probability of.

When S is finite (like now) we usually make Ω consist of all subsets of S.

What do these events mean?

$\phi \leftarrow$ represents no number appeared on the die when you rolled it

$\{2\} \leftarrow$ represents a 2 occured when you rolled the die

$\{1, 4\} \leftarrow$ represents either 1 or 4 occured when you roll the die

$\{2,3,4\} \leftarrow$ represents that either 2 or 3 or 4 occured

$\{1,2,3,4\} \leftarrow$ represents that either 1 or 2 or 3 or 4 occured

Now we make a probability function $P : \Omega \longrightarrow \mathbb{R}$

Let's assume each side of the die is equally likely.

First assign

$$P(\phi) = 0$$

Then assign the probability of each outcome.

$$\left.\begin{aligned} P(\{1\}) &= 1/4 \\ P(\{2\}) &= 1/4 \\ P(\{3\}) &= 1/4 \\ P(\{4\}) &= 1/4 \end{aligned}\right\} \text{these add up to 1}$$

Now we extend P across all of Ω by doing disjoint sums. For example,

$$\begin{aligned} P(\{2,4\}) &= P(\{2\}) + P(\{4\}) \\ &= \frac{1}{4} + \frac{1}{4} \\ &= \frac{1}{2} \end{aligned}$$

$$P(\{1,2,3\}) = P(\{1\}) + P(\{2\}) + P(\{3\})$$
$$= \frac{1}{4} + \frac{1}{4} + \frac{1}{4}$$
$$= \frac{3}{4}$$

$$P(\{1,2,3,4\}) = P(\{1\}) + P(\{2\}) + P(\{3\}) + P(\{4\})$$
$$= \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}$$
$$= 1$$

Def: A **probability space** consists of two sets and a function (S, Ω, P).

S is called the **sample space** of our experiment. The elements of S are called the **outcomes** of the experiment.

Ω is a set whose elements are subsets of S. The elements of Ω are called **events**.

$P : \Omega \to \mathbb{R}$ is a function where for each E in Ω we get a probability $P(E)$.

Furthermore, the following

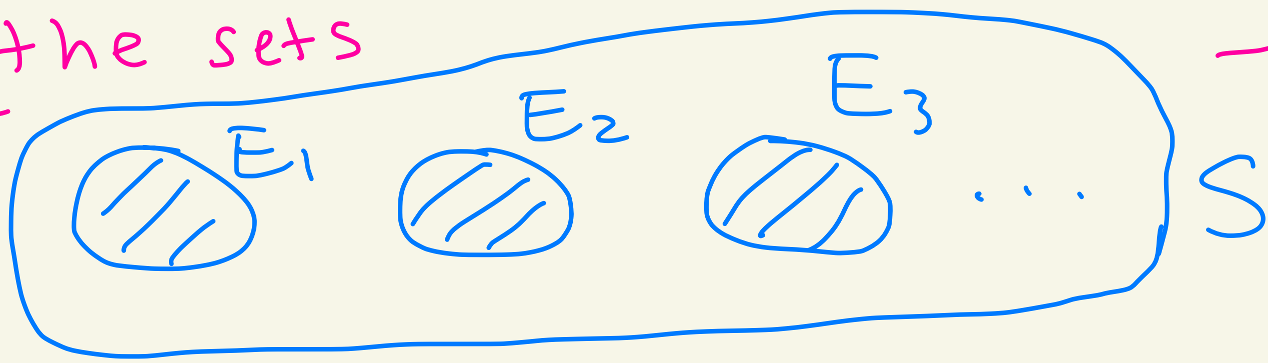
properties must hold:

1. S is an element of Ω.
   [you want to be able to measure $P(S)$]
2. If E is an event in Ω, then $\bar{E}$ is an event in Ω.
3. If $E_1, E_2, E_3, \ldots$ is a finite or infinite sequence of events from Ω, then $\bigcup_i E_i$ is in Ω.
4. $0 \leq P(E) \leq 1$ for all events E in Ω

⑤ $P(S) = 1$

⑥ If $E_1, E_2, E_3, \dots$ is a finite or infinite sequence of events from Ω that are pair-wise disjoint

<u>means:</u> $E_i \cap E_j = \phi$ if $i \neq j$
ie there is no overlap in the sets



then

$$\underbrace{P\left(\bigcup_i E_i\right)}_{P(E_1 \cup E_2 \cup E_3 \cup \dots)} = \underbrace{\sum_i P(E_i)}_{P(E_1)+P(E_2)+P(E_3)+\dots}$$

$P(E_1 \cup E_2 \cup E_3 \cup \dots) = P(E_1) + P(E_2) + P(E_3) + \dots$

$\boxed{\text{end of def}}$

This def is based on the work of Andrey Kolmogorov 1930s

Remark: A set Ω satisfying ①, ②, ③ above is called a <u>σ-algebra</u> or <u>σ-field</u>

Remark: If Ω is a σ-algebra one can show that

(a) $\phi \in \Omega$

(b) If $E_1, E_2, E_3, \ldots$ are in Ω, then $\bigcap_i E_i$ is in Ω.

Look at online notes for proof.

How to construct a probability space when S is finite

Suppose S is a finite sample space.

Define Ω to be the set of all subsets of S.

For each outcome ω in S pick a real number $0 \le n_w \le 1$ and define $P(\{w\}) = n_w$

Ex:

$S = \{1, 2, 3, 4\}$

$P(\{1\}) = 1/4 = n_1$ $\quad P(\{3\}) = 1/4 = n_3$

$P(\{2\}) = 1/4 = n_2$ $\quad P(\{4\}) = 1/4 = n_4$

At the same time pick the numbers so that

$$\underbrace{\sum_{w \in S}}_{\text{means: sum over all } w \text{ in } S} P(\{w\}) = 1$$

Ex above
$n_1 + n_2 + n_3 + n_4$
$= \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}$
$= 1$

Now extend P to all of Ω as follows:
If $E = \{w_1, w_2, \ldots, w_r\}$ is an event in Ω, then define

$$P(E) = \sum_{i=1}^{r} P(\{w_i\})$$

$$= P(\{w_1\}) + P(\{w_2\}) + \cdots + P(\{w_r\})$$

Ex: $P(\{1,2\}) = P(\{1\}) + P(\{2\})$

If $E = \emptyset$, define $P(\emptyset) = 0$.

<u>Theorem</u>: The above construction is a probability space.

See proof in notes.