Math 4740
9/30/24

M	W
9/30 Topic 3	10/2 Topic 3
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Theorem: Let (S, Ω, P) be a probability space.

① Let A and B be events and $P(A) > 0$. Then,

$$P(A \cap B) = P(A) \cdot P(B \mid A)$$

② Let $A_1, A_2, \ldots, A_n$ be events with $P(A_1 \cap A_2 \cap \cdots \cap A_n) > 0$.

Then,

$$P(A_1 \cap A_2 \cap \cdots \cap A_n)$$
$$= P(A_1) \cdot P(A_2 \mid A_1) \cdot P(A_3 \mid A_1 \cap A_2)$$
$$\cdot P(A_4 \mid A_1 \cap A_2 \cap A_3) \cdots$$
$$\cdots P(A_n \mid A_1 \cap A_2 \cap \cdots \cap A_{n-1})$$

③ (law of total probability)

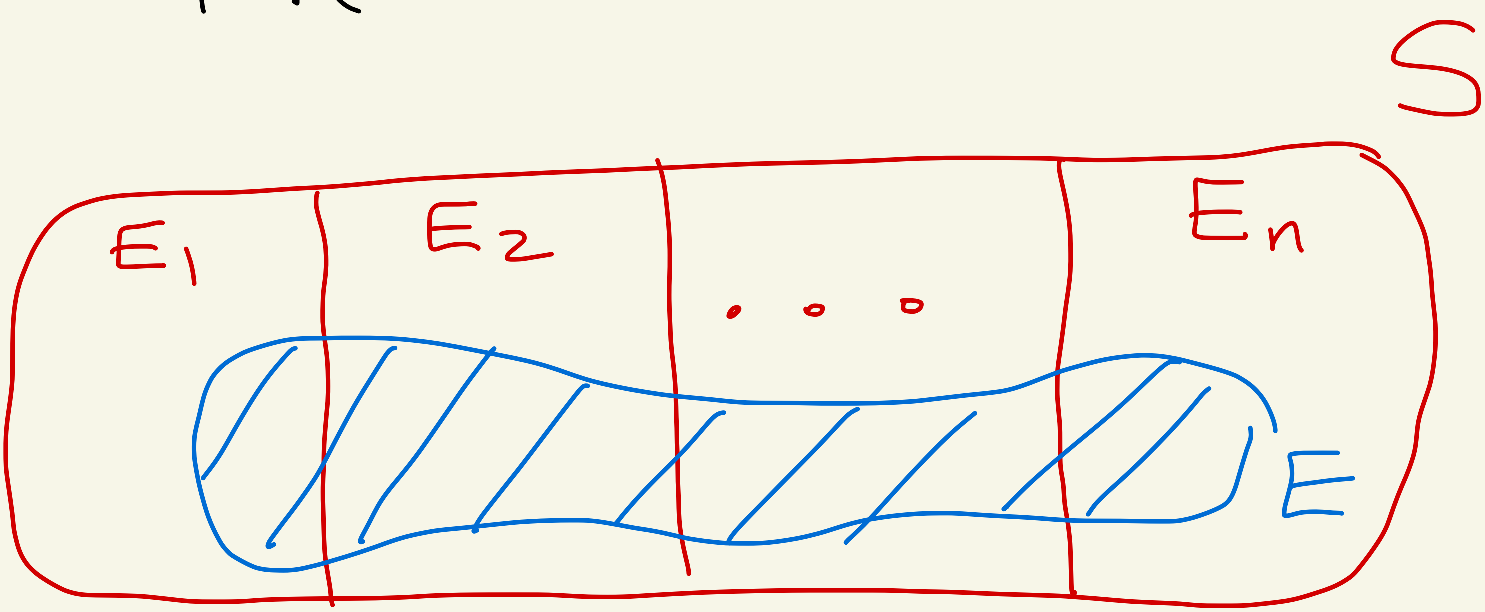
Suppose $S = E_1 \cup E_2 \cup \cdots \cup E_n$ where each $E_i \neq \emptyset$, and $E_i \cap E_j = \emptyset$ if $i \neq j$, and $P(E_i) \neq 0$ for each i.

S is broken into n disjoint events

Then, for any event E we have

$$
\begin{aligned}
P(E) = \; & P(E \mid E_1) \cdot P(E_1) \\
& + P(E \mid E_2) \cdot P(E_2) \\
& + \cdots \\
& + P(E \mid E_n) \cdot P(E_n)
\end{aligned}
$$

$$
\begin{aligned}
& P(E \cap E_1) \\
& + P(E \cap E_2) \\
& + \cdots \\
& + P(E \cap E_n)
\end{aligned}
$$



Ex: Suppose there are three boxes.
In box 1 are two 4-sided dice.
In box 2 are two 6-sided dice.
In box 3 are two 8-sided dice.

Suppose you randomly pick a box (each box is equally likely) then take out the dice and roll them.
What's the probability that the sum of the dice is 8?

$\boxed{\text{box 1 picked}}$: $\boxed{\frac{\text{sum is 8}}{\{(4,4)\}}}$ → $\boxed{\frac{\text{probability}}{1/16}}$

box 2 picked:

sum is 8		probability
$\{(2,6),(3,5),(4,4),(5,3),(6,2)\}$	$\rightarrow$	$5/36$

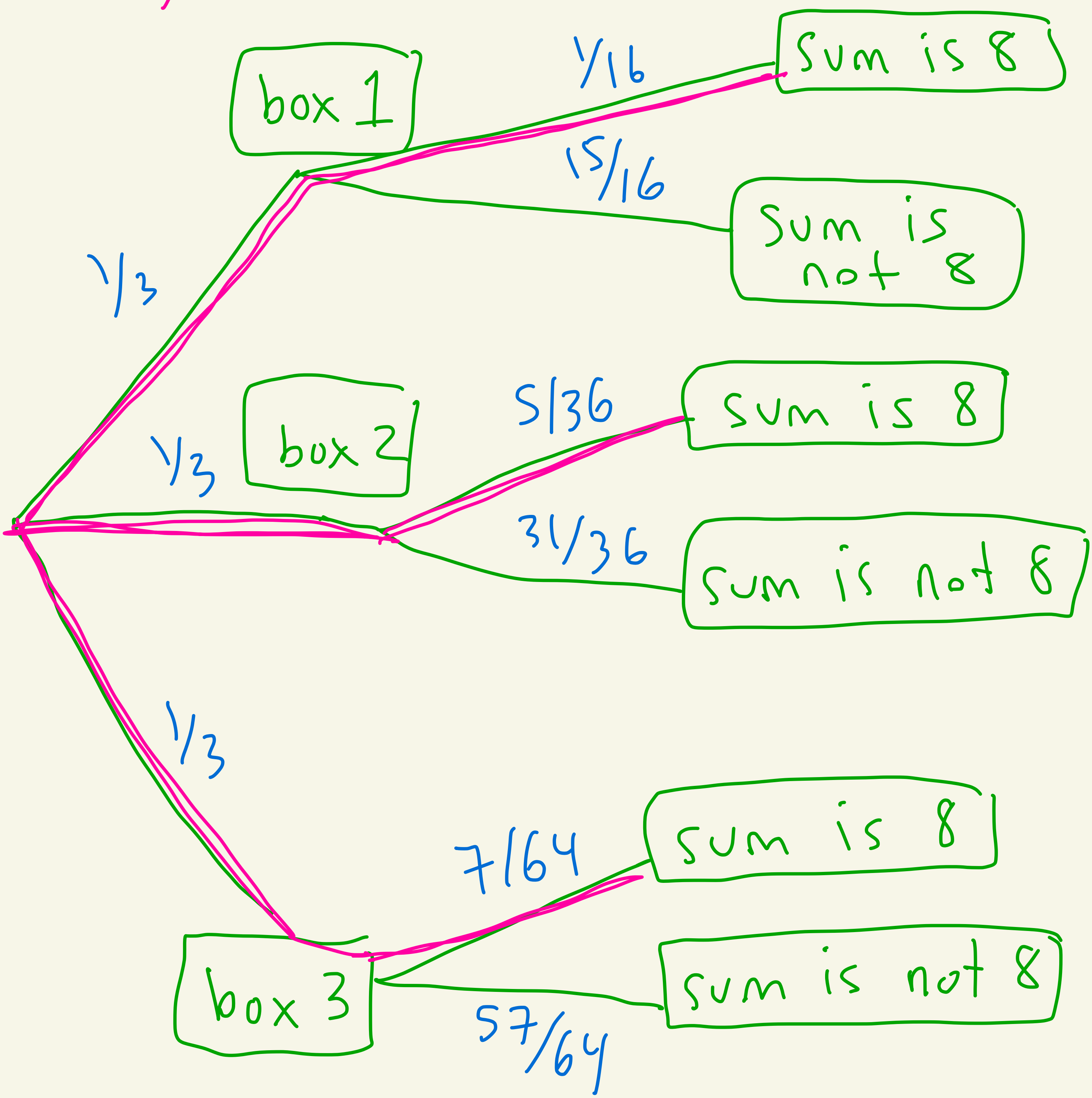
box 3 picked:

sum is 8		probability
$\{(1,7),(2,6),(3,5),(4,4),(5,3),(6,2),(7,1)\}$	$\rightarrow$	$7/64$

$$
\begin{aligned}
P(\text{sum is } 8) &= P(\text{sum is } 8 \mid \text{box 1 picked}) \cdot P(\text{box 1 picked}) \\
&+ P(\text{sum is } 8 \mid \text{box 2 picked}) \cdot P(\text{box 2 picked}) \\
&+ P(\text{sum is } 8 \mid \text{box 3 picked}) \cdot P(\text{box 3 picked})
\end{aligned}
$$

$$= \left(\frac{1}{16}\right)\left(\frac{1}{3}\right) + \left(\frac{5}{36}\right)\left(\frac{1}{3}\right) + \left(\frac{7}{64}\right)\left(\frac{1}{3}\right)$$

$$= \frac{11,456}{110,592} \approx \boxed{0.1036} \approx \boxed{10.36\%}$$



Add all the ways:

$$\left(\frac{1}{3}\right)\left(\frac{1}{16}\right) + \left(\frac{1}{3}\right)\left(\frac{5}{36}\right) + \left(\frac{1}{3}\right)\left(\frac{7}{64}\right)$$

<u>Ex:</u> (Monty Hall)

Let's redo the probability of the switch strategy from Monty Hall (start with door 1 and switch when Monty asks).

$$\begin{aligned} &P(\text{win car}) \\ &= P\left(\text{win car} \,\middle|\, \text{car behind door 1}\right) \cdot P\left(\text{car behind door 1}\right) \\ &\quad + P\left(\text{win car} \,\middle|\, \text{car behind door 2}\right) \cdot P\left(\text{car behind door 2}\right) \\ &\quad + P\left(\text{win car} \,\middle|\, \text{car behind door 3}\right) \cdot P\left(\text{car behind door 3}\right) \end{aligned}$$

$$= (0)\left(\frac{1}{3}\right) + (1)\left(\frac{1}{3}\right) + (1)\left(\frac{1}{3}\right)$$

$$= \frac{2}{3}$$

Given two events E and F sometimes we get

$$\underbrace{P(E|F)}_{\text{probability of E given F occured}} = P(E)$$

this eqn is saying if F occurs, it doesn't change the probability that E will occur

and sometimes not

When will $P(E|F) = P(E)$?

When

$$\frac{P(E\cap F)}{P(F)} = P(E)$$

That is, when

$$P(E\cap F) = P(E)\cdot P(F)$$

<u>Def:</u> We say that two events E and F are <u>independent</u> if

$$P(E\cap F) = P(E)\cdot P(F)$$

otherwise we say they are <u>dependent</u>.

<u>Note:</u> If $P(E)>0$, $P(F)>0$, then

E and F are independent

is equivalent to

$$\boxed{P(E \cap F) = P(E)\,P(F)}$$

is equivalent to

$$\boxed{\frac{P(E \cap F)}{P(E)} = P(F) \quad \text{OR} \quad \frac{P(E \cap F)}{P(F)} = P(E)}$$

is equivalent to

$$\boxed{P(F \mid E) = P(F) \quad \text{OR} \quad P(E \mid F) = P(E)}$$

HW 2

You roll ten 6-sided dice. What's the probability you get exactly one 4, six 5's, and the other three rolls aren't 4's or 5's?

Ex:

1 5 4 5 5 1 5 2 5 5

Sample space size:

$$6 \cdot 6 \cdot 6 \cdots 6 = 6^{10}$$

Count the outcomes w/ exactly one 4 and six 5's.

Step 1:

pick where the 4 goes

$$\_\ \_\ \_\ \underline{4}\ \_\ \_\ \_\ \_\ \_\ \_$$

$$\#\text{ ways} = \binom{10}{1} = 10$$

Step 2:

pick where the six 5's go.

$$\underbrace{\underline{5}\ \underline{5}\ \_\ \underline{4}\ \_\ \underline{5}\ \underline{5}\ \_\ \underline{5}}_{\text{out of the 9 open spots pick 6 of them}}\ \underline{5}$$

$$\#\text{ ways} = \binom{9}{6} = 84$$

Step 3:

Fill in the remaining 3 spots with 1, 2, 3, 6.

$$\underline{5}\ \underline{5}\ \underline{1}\ \underline{4}\ \underline{6}\ \underline{5}\ \underline{5}\ \underline{3}\ \underline{5}\ \underline{5}$$

$$4 \cdot 4 \cdot 4$$

$$\#\text{ ways} = 4^3 = 64$$

Answer $= \dfrac{10 \cdot 84 \cdot 64}{6^{10}} \approx 0.000889 \approx 0.0889\%$