Math 4740

HW 1 Solutions

1(a) $S = \{ (1,1), (1,2), (1,3), (1,4), (1,5), (1,6),$
$(2,1), (2,2), (2,3), (2,4), (2,5), (2,6),$
$(3,1), (3,2), (3,3), (3,4), (3,5), (3,6),$
$(4,1), (4,2), (4,3), (4,4), (4,5), (4,6),$
$(5,1), (5,2), (5,3), (5,4), (5,5), (5,6),$
$(6,1), (6,2), (6,3), (6,4), (6,5), (6,6) \}$

$P(\{(a,b)\}) = \frac{1}{36}$ for any (a,b) in S. That is, each element of S is equally weighted. If E is an event then $P(E) = \frac{|E|}{|S|} = \frac{|E|}{36}$

1(b) $A = \{(1,1)\}$
$B = \{(1,3), (2,2), (3,1)\}$

1(c) $A \cup B = \{(1,1), (1,3), (2,2), (3,1)\}$
$A \cap B = \phi$
$\overline{A} = \{$ all elements of S except $(1,1) \}$
$= \{ (1,2), (1,3), (1,4), (1,5), (1,6),$
$(2,1), (2,2), (2,3), (2,4), (2,5), (2,6),$
$(3,1), (3,2), (3,3), (3,4), (3,5), (3,6),$
$(4,1), (4,2), (4,3), (4,4), (4,5), (4,6),$
$(5,1), (5,2), (5,3), (5,4), (5,5), (5,6),$
$(6,1), (6,2), (6,3), (6,4), (6,5), (6,6) \}$

$\overline{B} = \{$ all elements of S except $(1,3),(2,2),(3,1)\}$

$$
\begin{aligned}
= \{ & (1,1), (1,2), (1,4), (1,5), (1,6), \\
& (2,1), (2,3), (2,4), (2,5), (2,6), \\
& (3,2), (3,3), (3,4), (3,5), (3,6), \\
& (4,1), (4,2), (4,3), (4,4), (4,5), (4,6), \\
& (5,1), (5,2), (5,3), (5,4), (5,5), (5,6), \\
& (6,1), (6,2), (6,3), (6,4), (6,5), (6,6) \}
\end{aligned}
$$

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1 (d)

$A \cup B$ consists of all the rolls where the sum of the dice is either 2 or 4

$A \cap B$ consists of all the rolls where the sum of the dice is 2 and 4 at the same time which is impossible, hence $A \cap B$ is empty

$\overline{A}$ consists of all the rolls of the dice where the sum is not 2

$\overline{B}$ consists of all the rolls of the dice where the sum is not 4

1 (e)

$$P(A) = \frac{1}{36}$$

$$P(B) = \frac{3}{36}$$

$$P(A \cup B) = \frac{4}{36}$$

$$P(A \cap B) = 0$$

$$P(\overline{A}) = \frac{35}{36}$$

$$P(\overline{B}) = \frac{33}{36}$$

② (a)

$$
\begin{aligned}
S = \{ & (1,1), (1,2), (1,3), (1,4), (1,5), (1,6), \\
& (2,1), (2,2), (2,3), (2,4), (2,5), (2,6), \\
& (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), \\
& (4,1), (4,2), (4,3), (4,4), (4,5), (4,6), \\
& (5,1), (5,2), (5,3), (5,4), (5,5), (5,6), \\
& (6,1), (6,2), (6,3), (6,4), (6,5), (6,6) \}
\end{aligned}
$$

$P(\{(a,b)\}) = \frac{1}{36}$ for any (a,b) in S. That is, each element of S is equally weighted. If E is an event then $P(E) = \frac{|E|}{|S|} = \frac{|E|}{36}$

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② (b)

$$
\begin{aligned}
A = \{ & (1,2), (1,4), (1,6), \\
& (2,1), (2,2), (2,3), (2,4), (2,5), (2,6), \\
& (3,2), (3,4), (3,6), \\
& (4,1), (4,2), (4,3), (4,4), (4,5), (4,6), \\
& (5,2), (5,4), (5,6), \\
& (6,1), (6,2), (6,3), (6,4), (6,5), (6,6) \}
\end{aligned}
$$

$$
\begin{aligned}
B = \{ & (2,2), (2,4), (2,6), (4,2), (4,4), (4,6), \\
& (6,2), (6,4), (6,6) \}
\end{aligned}
$$

$C = \{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),$
$(2,1),(2,3),(2,5)$
$(3,1),(3,2),(3,3),(3,4),(3,5),(3,6),$
$(4,1),(4,3),(4,5),$
$(5,1),(5,2),(5,3),(5,4),(5,5),(5,6),$
$(6,1),(6,3),(6,5)\}$

$D = \{(1,1),(1,3),(1,5),(3,1),(3,3),(3,5),$
$(5,1),(5,3),(5,5)\}$

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2 (c)

$A \cap C = \{(1,2),(1,4),(1,6),(2,1),(2,3),(2,5),$
$(3,2),(3,4),(3,6),(4,1),(4,3),(4,5),$
$(5,2),(5,4),(5,6),(6,1),(6,3),(6,5)\}$

$\bar{A} = D$

$B \cap D = \phi$

$B \cup D = \{(1,1),(1,3),(1,5),(2,2),(2,4),(2,6),$
$(3,1),(3,3),(3,5),(4,2),(4,4),(4,6),$
$(5,1),(5,3),(5,5),(6,2),(6,4),(6,6)\}$

$\bar{B} = C$

$\bar{D} = A$

② (d)

$A \cap C$ are the dice rolls where one of the dice is even and one of the dice is odd.

$\bar{A}$ are the dice rolls where both dice are odd.

$B \cup D$ are the dice rolls where either both dice are even or both dice are odd.

$B \cap D$ is empty since it consists of the dice rolls where both dice are even <u>and</u> both dice are odd, which can't happen

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② (e)

$P(A) = \frac{27}{36}$ $\quad P(B) = \frac{9}{36}$ $\quad P(C) = \frac{27}{36}$

$P(\bar{A}) = \frac{9}{36}$

$P(D) = \frac{9}{36}$ $\quad P(A \cap C) = \frac{18}{36}$

$P(\bar{B}) = \frac{27}{36}$

$P(B \cup D) = \frac{18}{36}$

$P(B \cap D) = 0$

$P(\bar{D}) = \frac{27}{36}$

③ (a)

$$S = \{(H,H,H,H), (H,H,H,T), (H,H,T,H), (H,H,T,T),$$
$$(H,T,H,H), (H,T,H,T), (H,T,T,H), (H,T,T,T),$$
$$(T,H,H,H), (T,H,H,T), (T,H,T,H), (T,H,T,T),$$
$$(T,T,H,H), (T,T,H,T), (T,T,T,H), (T,T,T,T)\}$$

$P(\{(a,b,c,d)\}) = \frac{1}{16}$ for any (a,b,c,d) in S.

That is the elements are equally weighted.

If E is an event then

$$P(E) = \frac{|E|}{|S|} = \frac{|E|}{16}$$

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③ (b)

$$A = \{(H,H,H,H), (H,H,H,T), (H,H,T,H), (H,H,T,T),$$
$$(H,T,H,H), (H,T,H,T), (H,T,T,H), (H,T,T,T)\}$$

$$B = \{(H,T,H,T), (H,T,T,T), (T,T,H,T), (T,T,T,T)\}$$

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③ (c)

$$A \cup B = \{(H,H,H,H), (H,H,H,T), (H,H,T,H), (H,H,T,T), (H,T,H,H), (H,T,H,T), (H,T,T,H), (H,T,T,T), (T,T,H,T), (T,T,T,T)\}$$

$$A \cap B = \{(H,T,H,T), (H,T,T,T)\}$$

$$\overline{A} = \{(T,H,H,H), (T,H,H,T), (T,H,T,H), (T,H,T,T), (T,T,H,H), (T,T,H,T), (T,T,T,H), (T,T,T,T)\}$$

$$\overline{B} = \{(H,H,H,H), (H,H,H,T), (H,H,T,H), (H,H,T,T), (H,T,H,H), (H,T,T,H), (T,H,H,H), (T,H,H,T), (T,H,T,H), (T,H,T,T), (T,T,H,H), (T,T,T,H)\}$$

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③ (d)

$A \cup B$ are the elements that either begin with a head on the first flip OR have a tails on the second or fourth flip.

$A \cap B$ are the elements with heads on the first flip and tails on the second and fourth flips

$\overline{A}$ are the elements that have a tails on the first flip.

$\overline{B}$ are the elements where either the second or the fourth flip is a heads.

③ (e)

$P(A) = \frac{8}{16}$

$P(B) = \frac{4}{16}$

$P(A \cap B) = \frac{2}{16}$

$P(A \cup B) = \frac{10}{16}$

Note that
$P(A \cup B) \neq P(A) + P(B)$
because $A \cap B \neq \emptyset$.
Could use the formula
$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(\bar{A}) = \frac{8}{16}$

$P(\bar{B}) = \frac{12}{16}$

④ (a) $S = \{1,2,3,4\}$

$P(\{1\}) = \frac{2}{8}$ $\quad$ $P(\{2\}) = \frac{2}{8}$

$P(\{3\}) = \frac{3}{8}$ $\quad$ $P(\{4\}) = \frac{1}{8}$

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④ (b) $A = \{1,3\}$

$P(A) = P(\{1,3\}) = P(\{1\}) + P(\{3\}) = \frac{2}{8} + \frac{3}{8} = \frac{5}{8}$

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④ (c) $B = \{1,2,3\}$

$P(B) = P(\{1,2,3\}) = P(\{1\}) + P(\{2\}) + P(\{3\})$

$= \frac{2}{8} + \frac{2}{8} + \frac{3}{8} = \frac{7}{8}$

Note: In this problem the elements of S are not equally weighted, so it is not true that $P(E) = \frac{|E|}{|S|}$.

5

$S = \{(1,1),(1,2),(1,3),(1,4),(2,1),(2,2),(2,3),(2,4),$
$(3,1),(3,2),(3,3),(3,4),(4,1),(4,2),(4,3),(4,4)\}$

$P(\{(a,b)\}) = \frac{1}{16}$ for each (a,b) in S,

that is the elements are equally weighted.

So, $P(E) = \frac{|E|}{|S|} = \frac{|E|}{16}$ for any event E.

5(a) Let A be the event that at least one of the dice shows a 2. Then,

$A = \{(1,2),(2,1),(2,2),(2,3),(2,4),(3,2),(4,2)\}$

$P(A) = \frac{7}{16}$

5(b) Let B be the event that the sum of the dice is a 4.

$B = \{(1,3),(2,2),(3,1)\}$ $P(B) = \frac{3}{16}$

5(c) Let C be the event that the sum of the dice is either 5 or 7.

$C = \{(1,4),(4,1),(2,3),(3,2),(3,4),(4,3)\}$

$P(C) = \frac{6}{16}$

⑥ The sample space is

$$S = \{(g,r) \mid g,r = 1,2,3,4,5,6,7,8\}$$

$$= \{ (1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (1,7), (1,8),$$
$$(2,1), (2,2), (2,3), (2,4), (2,5), (2,6), (2,7), (2,8),$$
$$(3,1), (3,2), (3,3), (3,4), (3,5), (3,6), (3,7), (3,8),$$
$$(4,1), (4,2), (4,3), (4,4), (4,5), (4,6), (4,7), (4,8),$$
$$(5,1), (5,2), (5,3), (5,4), (5,5), (5,6), (5,7), (5,8),$$
$$(6,1), (6,2), (6,3), (6,4), (6,5), (6,6), (6,7), (6,8),$$
$$(7,1), (7,2), (7,3), (7,4), (7,5), (7,6), (7,7), (7,8),$$
$$(8,1), (8,2), (8,3), (8,4), (8,5), (8,6), (8,7), (8,8)\}$$

All the elements are equally weighted so if E is an event then $P(E) = \frac{|E|}{|S|} = \frac{|E|}{64}$

Let A be the event that the red die has a larger value than the green die. Then,

$$A = \{(1,2),(1,3),(1,4),(1,5),(1,6),(1,7),(1,8),$$
$$(2,3),(2,4),(2,5),(2,6),(2,7),(2,8),$$
$$(3,4),(3,5),(3,6),(3,7),(3,8),$$
$$(4,5),(4,6),(4,7),(4,8),$$
$$(5,6),(5,7),(5,8),$$
$$(6,7),(6,8),$$
$$(7,8)\}$$

Thus, $P(A) = \dfrac{|A|}{|S|} = \dfrac{28}{64}$

⑦ $S = \{ W, R, G \}$

$P(\{W\}) = \frac{1}{3}$ $\quad P(\{R\}) = \frac{1}{3}$ $\quad P(\{G\}) = \frac{1}{3}$

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⑧ $S = \{ \{W,R\}, \{W,G\}, \{R,G\} \}$

There are 3 elements of S.

We use sets $\{W,R\}$ for example instead of (W,R) because we don't care about the order. And $(W,R) \neq (R,W)$ but $\{W,R\} = \{R,W\}$.

$\{W,R\}$ represents choosing 1 white and 1 red ball.

$\{W,G\}$ represents choosing 1 white and 1 green ball.

$\{R,G\}$ represents choosing 1 red and 1 green ball.

$P(\{W,R\}) = \frac{1}{3}$ $\quad P(\{W,G\}) = \frac{1}{3}$ $\quad P(\{R,G\}) = \frac{1}{3}$

⑨ Define

$$S = \{ \underbrace{(2)}_{\text{2 on first roll}}, \underbrace{(1,2), (3,2), (4,2)}_{\text{2 on 2nd roll}},$$

$$\underbrace{(1,1,2), (1,3,2), (1,4,2), (3,1,2), (3,3,2), (3,4,2), (4,1,2), (4,3,2), (4,4,2)}_{\text{2 on 3rd roll}}, \underbrace{\ldots}_{\text{2 on 4th, 5th, 6th, ... roll}} \}$$

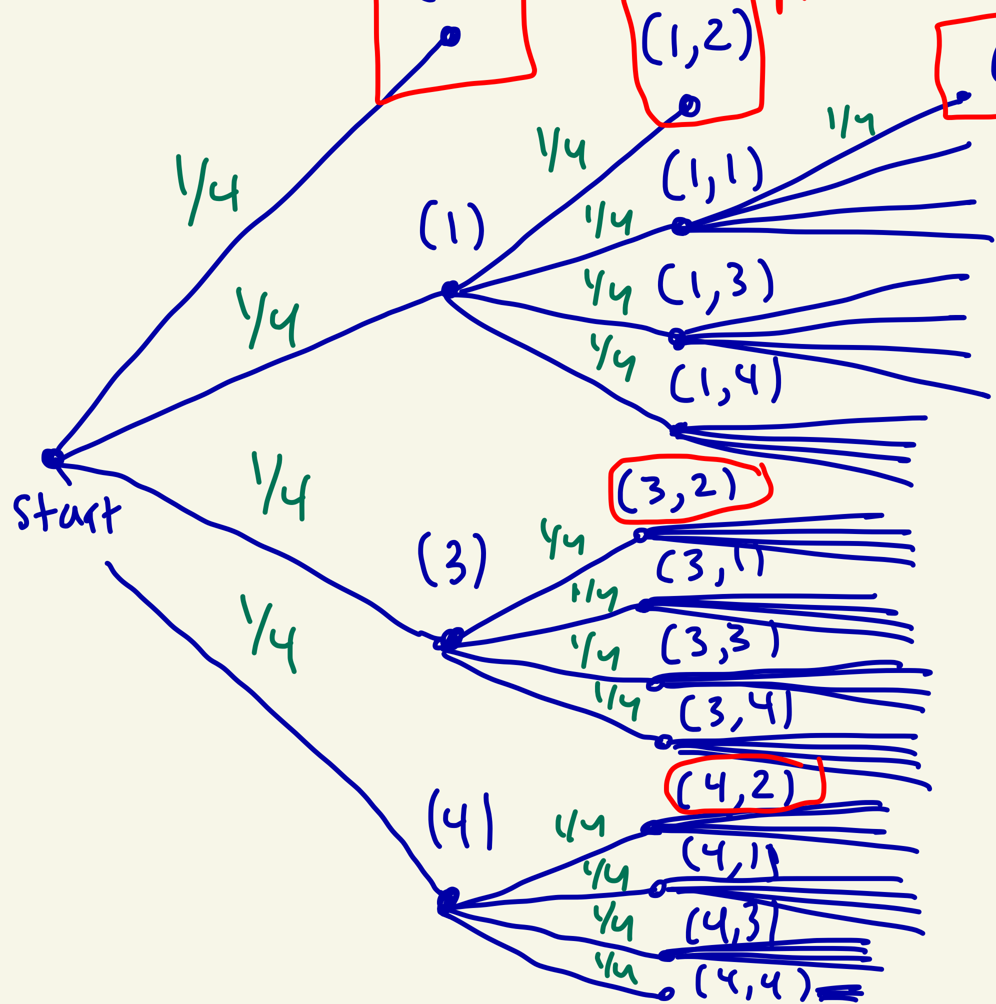
$$= \{(2)\} \cup \{ (a_1, a_2, \ldots, a_n, 2) \mid a_1, a_2, \ldots, a_n = 1,3,4 \text{ and } n \geq 1 \}$$

To define P think of the experiment this way

probability is $\frac{1}{4}$

probability is $\left(\frac{1}{4}\right)^2$

probability is $\left(\frac{1}{4}\right)^3$



Tree keeps going forever except stops at each red spot with probabilities in green

Define

$P(\{(2)\}) = \frac{1}{4}$

$P(\{(a_1, 2)\}) = \left(\frac{1}{4}\right)^2$ where $a_1 = 1, 3, 4$

$P(\{(a_1, a_2, 2)\}) = \left(\frac{1}{4}\right)^3$ where $a_1, a_2 = 1, 3, 4$

and in general

$P(\{(a_1, a_2, \ldots, a_n, 2)\}) = \left(\frac{1}{4}\right)^{n+1}$.

For example, $P((1,2)) = \left(\frac{1}{4}\right)^2$ and $P((1,3,2)) = \left(\frac{1}{4}\right)^3$.

Let Ω be all subsets of S.

If $E \in \Omega$, define $P(E) = \sum_{w \in E} P(\{w\})$.

We see that $0 \le P(\{w\}) \le 1$ for any $w \in S$. Now lets show $P(S) = 1$.

We have that

$$P(S) = \sum_{w \in S} P(\{w\})$$

$$= P(\{(2)\})$$

$$+ P(\{(1,2)\}) + P(\{(3,2)\}) + P(\{(4,2)\})$$

$$+ P(\{(1,1,2)\}) + P(\{(1,3,2)\}) + P(\{(1,4,2)\})$$

$$+ P(\{(3,1,2)\}) + P(\{(3,3,2)\}) + P(\{(3,4,2)\})$$

$$+ P(\{(4,1,2)\}) + P(\{(4,3,2)\}) + P(\{(4,4,2)\})$$

$$+ \cdots$$

$$= \frac{1}{4} + 3 \cdot \left(\frac{1}{4}\right)^2 + 3^2 \cdot \left(\frac{1}{4}\right)^3$$

$$+ 3^3 \cdot \left(\frac{1}{4}\right)^4 + 3^4 \cdot \left(\frac{1}{4}\right)^5 + \cdots$$

$$= \frac{1}{4}\left[1 + \left(\frac{3}{4}\right) + \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^3 + \cdots\right]$$

$$= \frac{1}{4}\left[\frac{1}{1-\frac{3}{4}}\right]$$

$$= \frac{1}{4}[4] = 1.$$

Calculus

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$$

if $|x| < 1$

Thus, $P(S) = 1$.

Since Ω is all subsets of S we know that Ω satisfies axioms 1, 2, 3 of the def of probability space.

We just verified 4, 5 axioms for P.

Axiom 6 is true since we defined

$$P(E) = \sum_{w \in E} P(\{w\}).$$

10 See the solution for #9 before reading this. We do the same idea, but I put more details in the solution for 9.

Define

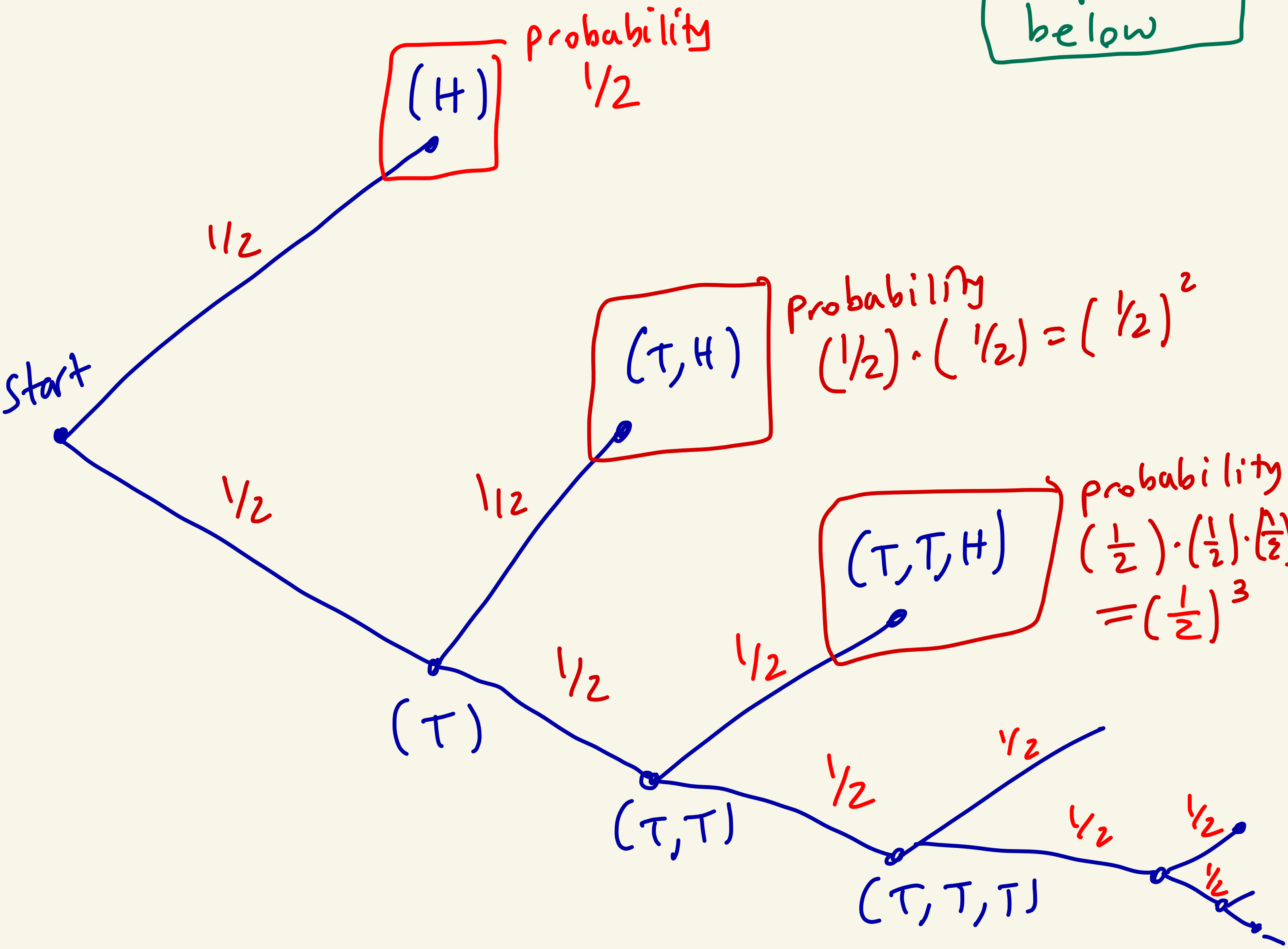
$$S = \{ (H), (T,H), (T,T,H), (T,T,T,H), (T,T,T,T,H), \ldots \}$$

Let Ω be all subsets of S.

Define

$$P(\{(\underbrace{T,T,\dots,T}_{k \text{ tails}},H)\}) = \underbrace{\left(\tfrac{1}{2}\right)^{k}\left(\tfrac{1}{2}\right) = \left(\tfrac{1}{2}\right)^{k+1}}$$

see picture below



Then for any event E define

$$P(E) = \sum_{\omega \in E} P(\{\omega\})$$

By how we defined Ω, Ω will satisfy axioms 1, 2, 3 of a probability space.

By how we defined P, P will satisfy axioms 4, 6 of a probability space.

Thus, we just need to verify axiom 5.

$$P(S) = \sum_{\omega \in S} P(\{\omega\})$$

$$= P(\{(H)\}) + P(\{(T,H)\}) + P(\{(T,T,H)\}) + P(\{(T,T,T,H)\}) + \dots$$

$$= \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4 + \dots$$

$$= \frac{1}{2}\left[1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \dots\right]$$

Calculus

$1 + x + x^2 + x^3 + \dots = \frac{1}{1-x}$

if $|x| < \frac{1}{2}$

$$= \frac{1}{2}\left[\frac{1}{1-\frac{1}{2}}\right] = \frac{1}{2}[2] = 1.$$