Math 4740

HW 2 Solutions

1 (a)

$$\underset{\substack{\uparrow\\ 9\\ \text{possibilities}}}{\underline{1-9}}\quad \underset{\substack{\uparrow\\ 10\\ \text{possibilities}}}{\underline{0-9}}\quad \underset{\substack{\uparrow\\ 10}}{\underline{0-9}}\quad \underset{\substack{\uparrow\\ 10}}{\underline{0-9}}\quad \underset{\substack{\uparrow\\ 10}}{\underline{0-9}}\quad \underset{\substack{\uparrow\\ 10}}{\underline{0-9}}$$

There are

$$9 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = \boxed{900{,}000}$$

six digit numbers

① (b) Instead we calculate all the six digit numbers <u>without</u> a 5 and subtract this from 1(a).

$$\underset{\substack{8\\ \text{possibilities}}}{\underline{\substack{1-9\\ \text{not } 5}}} \quad \underset{\substack{9\\ \text{possibilities}}}{\underline{\substack{0-9\\ \text{not } 5}}} \quad \underset{9}{\underline{\substack{0-9\\ \text{not } 5}}} \quad \underset{9}{\underline{\substack{0-9\\ \text{not } 5}}} \quad \underset{9}{\underline{\substack{0-9\\ \text{not } 5}}} \quad \underset{9}{\underline{\substack{0-9\\ \text{not } 5}}}$$

There are

$$\underbrace{900{,}000}_{\#\text{ six digit numbers}} - \underbrace{8 \cdot 9 \cdot 9 \cdot 9 \cdot 9 \cdot 9}_{\#\text{ six digit numbers without a } 5}$$

$$= 900{,}000 - 472{,}392$$

$$= \boxed{427{,}608}$$

six digit numbers without a 5.

2

A - Z	A - Z not same as spot 1	A - Z not same as spot 1 or 2	0 - 9	0 - 9 not same as spot 4	0 - 9 not same as spot 4 or 5
$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$
26 possibilities	25 possibilities	24 possibilities	10 possibilities	9 possibilities	8 possibilities

# of license plates is

$$26 \cdot 25 \cdot 24 \cdot 10 \cdot 9 \cdot 8 = \boxed{11{,}232{,}000}$$

(3) (a) There are 5 letters so there are $5! = 120$ permutations.

---

(3) (b)

a	b/d/e	b/d/e not same as spot 2	b/d/e not same as spot 2 or 3	c
	3 possibilities	2 possibilities	1 possibility	

There are $\boxed{3 \cdot 2 \cdot 1 = 6}$ possible permutations.

Here is the choice tree:

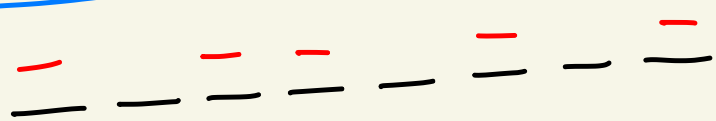
- a b _ _ c
  - a b d _ c — a b d e c
  - a b e _ c — a b e d c
- a d _ _ c
  - a d b _ c — a d b e c
  - a d e _ c — a d e b c
- a e _ _ c
  - a e b _ c — a e b d c
  - a e d _ c — a e d b c

(4) You need to put 5 dashes and 3 dots into $5+3=8$ spots.

The number of possible messages of this type can be calculated by picking the 5 spots amongst the 8 total spots where the dashes go.

This can be done in

$$\binom{8}{5} = \frac{8!}{3!\,5!} = \frac{8\cdot 7\cdot \cancel{6}\cdot \cancel{5!}}{\cancel{6}\cdot \cancel{5!}} = 56 \text{ ways.}$$

Example: 

Since the other spots have to be dots there is only one possible way to fill in the remaining spots with dots.

Thus, the answer is

$$56\cdot 1 = \boxed{56 \text{ possible messages}}$$

5 (a) Some examples are

01121120

11111111

12000121

Let's count!

$$\underset{\substack{\uparrow\\3}}{\overset{0/1/2}{\underline{\qquad}}}\quad\underset{\substack{\uparrow\\3}}{\overset{0/1/2}{\underline{\qquad}}}\quad\underset{\substack{\uparrow\\3}}{\overset{0/1/2}{\underline{\qquad}}}\quad\underset{\substack{\uparrow\\3}}{\overset{0/1/2}{\underline{\qquad}}}\quad\underset{\substack{\uparrow\\3}}{\overset{0/1/2}{\underline{\qquad}}}\quad\underset{\substack{\uparrow\\3}}{\overset{0/1/2}{\underline{\qquad}}}\quad\underset{\substack{\uparrow\\3}}{\overset{0/1/2}{\underline{\qquad}}}\quad\underset{\substack{\uparrow\\3}}{\overset{0/1/2}{\underline{\qquad}}}$$

possibilities

There are $3\cdot3\cdot3\cdot3\cdot3\cdot3\cdot3\cdot3 = 3^8$

$= \boxed{6561 \text{ possible sequences}}$

5 (b)

Step 1:

Pick 4 spots from the 8 total spots where the 0's go. This can be done in $\binom{8}{4} = \frac{8!}{4!4!}$

$= 70$ ways

One of the 70 ways:

0 0 _ 0 _ 0 _ _

Step 2:

Now there is no choice at this point, you must fill the remaining 4 spots with 1's. Thus, only 1 possibility at this step.

The above example becomes

0 0 1 0 1 0 1 1

Answer:

Total # of sequences is $70 \cdot 1 = \boxed{70}$

5 (c)

Step 1: Pick 3 spots from the 8 total spots to put the 0's. There are

$$\binom{8}{3} = \frac{8!}{5!\,3!} = \frac{8\cdot 7\cdot \cancel{6}\cdot \cancel{5!}}{\cancel{5!}\,\cancel{6}} = 56$$

ways to do this

Example possibility

$0\ \_\ 0\ 0\ \_\ \_\ \_\ \_$

Step 2: Pick 3 spots from the remaining 5 spots to put the 1's in. There are

$$\binom{5}{3} = \frac{5!}{2!\,3!} = \frac{5\cdot 4\cdot \cancel{3!}}{2\cdot \cancel{3!}} = 10$$

ways to do this

Example possibility

$0\ \_\ 0\ 0\ 1\ 1\ \_\ 1$

Step 3: Only 1 choice to make now: Fill the remaining spots with 2's

Example possibility

$0\ 2\ 0\ 0\ 1\ 1\ 2\ 1$

Answer: Total number of sequences is $56\cdot 10\cdot 1 = \boxed{560}$

6 (a) There are 13 books. Thus there are $13! = \boxed{6,227,020,800}$ ways to put them on the shelf

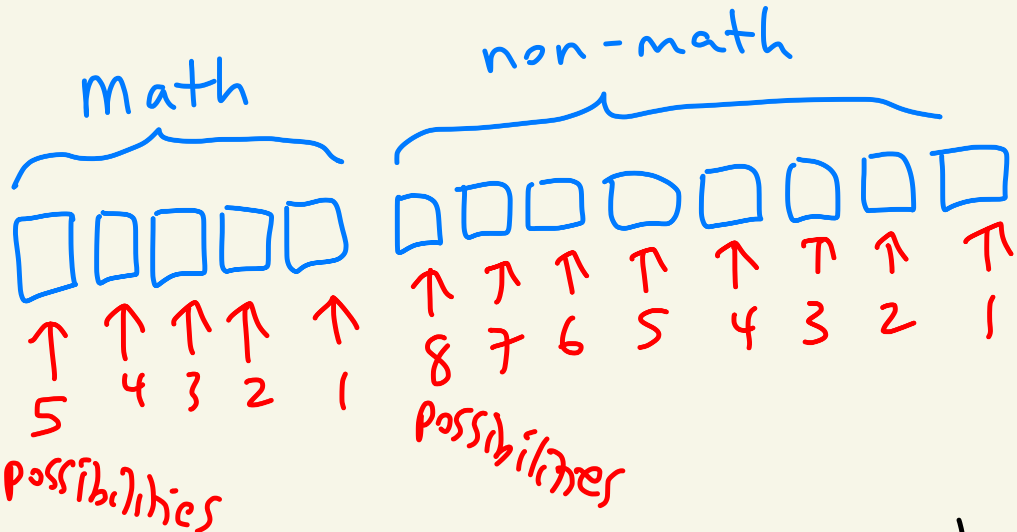
$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$
13	12	11	10	9	8	7	6	5	4	3	2	1
choices			choices									

⑥ (b) There are 5 math books and 8 other books. The math books have to be clumped together.

Step 1: Pick where the math books go. Think of the math books as one unit on this step

9 possibilities

Step 2: Now for each of the 9 possibilities from step 1 you must fill the books in in every possible permutation.



So step 2 gives $(5!)(8!)$

$$= (120)(40{,}320)$$

$$= 4{,}838{,}400 \text{ ways}$$

Thus, the total # of ways to put the books on the shelf with the math books next to each other is

$$(9)(5!)(8!) = (9)(4{,}838{,}400)$$

$$= \boxed{43{,}545{,}600 \text{ ways}}$$

7

There are $5+6=11$ people that can sit down in a row.

_	_	_	_	_	_	_	_	_	_	_
↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑
11	10	9	8	7	6	5	4	3	2	1
possibilities				possibilities						

So, there are $11! = 39{,}916{,}800$ ways the mathematicians and biologists can sit down.

Now we count the number of ways they can sit down with the mathematicians sitting together and the biologists sitting together.

Step 1: There are two possible templates.

M_1	M_2	M_3	M_4	M_5	B_1	B_2	B_3	B_4	B_5	B_6

OR

B_1	B_2	B_3	B_4	B_5	B_6	M_1	M_2	M_3	M_4	M_5

Step 2: Now fill in the spots.

M1	M2	M3	M4	M5	B1	B2	B3	B4	B5	B6
5	4	3	2	1	6	5	4	3	2	1
possibilities					possibilities					

This case gives $(5!)(6!)$ possibilities

OR

B1	B2	B3	B4	B5	B6	M1	M2	M3	M4	M5
6	5	4	3	2	1	5	4	3	2	1
possibilities						possibilities				

This case gives $(6!)(5!)$ possibilities

Adding these cases gives

$$(5!)(6!) + (6!)(5!) = 86{,}400 + 86{,}400$$

$$= 172{,}800 \text{ ways}$$

Thus the probability is

$$\boxed{\frac{172{,}800}{39{,}916{,}800} \approx 0.004329\ldots \approx 0.43\,\%}$$

(8) The sample space size is $|S| = 6^6 = 46{,}656$.

Let E be the event that at least two of the dice have the same number. We want $P(E)$.

Instead we will calculate $P(E) = 1 - P(\overline{E})$ where $\overline{E}$ is the event that none of the dice have the same number.

die 1	die 2	die 3	die 4	die 5	die 6
6 choices	5 choices since can't be the same as die 1	4 choices since can't be the same as die 1 or die 2	3 choices since can't be the same as die 1 or die 2 or die 3	2 choices since can't be the same as die 1 or die 2 or die 3 or die 4	1 choice since can't be the same as die 1 or die 2 or die 3 or die 4 or die 5

So, $P(\overline{E}) = \dfrac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{6^6} = \dfrac{720}{46{,}656}$

Thus, $P(E) = 1 - P(\overline{E}) = \dfrac{46{,}656 - 720}{46{,}656} = \boxed{\dfrac{45{,}936}{46{,}656}}$

$\approx \boxed{0.9845679\ldots}$

$\approx \boxed{98.46\%}$

(9) The sample space has size

$$|S| = 8 \cdot 8 \cdot 8 \cdot 8 = 8^4 = 4{,}096.$$

(a) First choose where the two 3's can go by picking two of the four spots.

$$\boxed{\binom{4}{2} = \frac{4!}{2!\,2!} = 6}$$

$\underline{3}$ $\underline{3}$ $\underline{\ \ }$ $\underline{\ \ }$
$\underline{3}$ $\underline{\ \ }$ $\underline{3}$ $\underline{\ \ }$
$\underline{3}$ $\underline{\ \ }$ $\underline{\ \ }$ $\underline{3}$
$\underline{\ \ }$ $\underline{3}$ $\underline{3}$ $\underline{\ \ }$
$\underline{\ \ }$ $\underline{3}$ $\underline{\ \ }$ $\underline{3}$
$\underline{\ \ }$ $\underline{\ \ }$ $\underline{3}$ $\underline{3}$

these are the 6 possibilities we are counting

Then fill in the remaining two spots with two numbers that aren't 3's.

ex: $\underline{3}$ $\underline{3}$ $\underset{\substack{\uparrow \\ 7 \\ \text{choices}}}{\underline{\ \ }}$ $\underset{\substack{\uparrow \\ 7 \\ \text{choices}}}{\underline{\ \ }}$

$$\boxed{7 \cdot 7 = 49}$$

There are $6 \cdot 49 = 294$ possibilities.

So, the probability is $\boxed{\dfrac{294}{4096}} \approx \boxed{0.07178\ldots}$

$\approx \boxed{7.18\,\%}$

(b)

$$P(\text{at most two 8's}) = P(\text{no 8's}) + P(\text{exactly one 8}) + P(\text{exactly two 8's})$$

$$= \frac{\overbrace{7\cdot 7\cdot 7\cdot 7}^{\text{fill in the four spots with \#s that aren't 8's}}}{4096}$$

$$+ \frac{\overbrace{\binom{4}{1}\cdot 7\cdot 7\cdot 7}^{\text{choose where the one 8 goes then fill in the remaining three spots with \#s that aren't 8's}}}{4096}$$

note:

$\binom{4}{1}=4$ is counting these 4 possibilities

8 _ _ _

_ 8 _ _

_ _ 8 _

_ _ _ 8

$$+ \frac{\overbrace{\binom{4}{2}\cdot 7\cdot 7}^{\text{choose where the two 8's go and then fill in the remaining two spots with \#s that aren't 8's}}}{4096}$$

Note

$\binom{4}{1}=4$

$\binom{4}{2}=6$

$$= \frac{2401+1372+294}{4096} = \boxed{\frac{4067}{4096}} \approx \boxed{0.9929\ldots} \approx \boxed{99.3\%}$$

(c)

$$P(\text{at least three 1's}) = P(\text{exactly three 1's}) + P(\text{exactly four 1's})$$

pick 3 spots out of the 4 spots for the 1's. Then fill the remaining spot with #s that aren't 1.

fill in all 4 spots with 1's there is only one way to do this

$$= \frac{\overbrace{\binom{4}{3}\cdot 7}}{4096} + \frac{\overbrace{\binom{4}{4}}}{4096}$$

$$= \frac{4\cdot 7}{4096} + \frac{1}{4096} = \boxed{\frac{29}{4096}} \approx \boxed{0.00708}$$

$$\approx \boxed{0.7\%}$$

(10)

The sample space has size $6^{10} = 60{,}466{,}176$

Now count possibilities

example possibility at this step

Step 1: Pick where the one 4 goes.

$\binom{10}{1} = 10$ possibilities

_ _ _ 4 _ _ _ _ _ _

example possibility at this step

Step 2: Pick where the six 5's go.

$$\binom{9}{6} = \frac{9!}{6!\,3!} = \frac{9\cdot 8\cdot 7\cdot 6!}{6!\,3!}$$

$$= \frac{9\cdot 8\cdot 7}{3!} =$$

$= 84$ possibilities

_ 5 5 4 5 _ 5 _ 5 5

example possibility at this step

Step 3: Fill in the other three spots with numbers that aren't 4 or 5.

$4\cdot 4\cdot 4 = 64$

possibilities

1 5 5 4 5 1 5 2 5 5

4 choices 4 choices 4 choice

The probability is thus

$$\frac{(10)(84)(64)}{60{,}466{,}176} = \boxed{\frac{53{,}760}{60{,}466{,}176}}$$

$$\approx \boxed{0.000889\ldots}$$

$$\approx \boxed{0.0889\ \%}$$

(11) The sample space has size $|S| = 2^5 = 32$

(a) Pick where the one head goes: $\binom{5}{1} = 5$

Fill in the remaining 4 spots with tails: $1 \cdot 1 \cdot 1 \cdot 1 = 1$

$$P(\text{exactly one head}) = \frac{5 \cdot 1}{32}$$

$$= \boxed{\frac{5}{32}} \approx \boxed{0.15625\ldots}$$

$$\approx \boxed{15.6\%}$$

We counted these possibilities:

H	T	T	T	T
T	H	T	T	T
T	T	H	T	T
T	T	T	H	T
T	T	T	T	H

(b) Pick where the three heads go: $\binom{5}{3} = \frac{5!}{3!\,2!} = 10$

Fill in the remaining 2 spots with tails: $1 \cdot 1 = 1$

$$P(\text{exactly three heads}) = \boxed{\frac{10}{32}} \approx \boxed{0.3125\ldots} \approx \boxed{31.25\%}$$

Note: The count of 10 above counted these:

H	H	H	T	T
H	H	T	H	T
H	H	T	T	H
H	T	H	H	T
H	T	H	T	H
H	T	T	H	H
T	H	H	H	T
T	H	H	T	H
T	H	T	H	H
T	T	H	H	H

(c) There is only 1 way to get all tails. It is $\underline{T}\ \underline{T}\ \underline{T}\ \underline{T}\ \underline{T}$

So,

$$p(\text{all tails}) = \boxed{\frac{1}{32}} \approx \boxed{0.03125} \approx \boxed{3.125\%}$$

(12) The sample space has size

$$|S| = 2^{20} = 1{,}048{,}576$$

(a)

$$P(\text{at least 2 heads}) = 1 - P(\text{less than 2 heads})$$

$$= 1 - P(\text{exactly 0 heads}) - P(\text{exactly 1 head})$$

only 1 way to have 0 heads. Fill all 20 spots with tails

exactly 1 head. Pick the spot where the head goes in $\binom{20}{1} = 20$ ways. Then fill the remaining spots with tails in 1 way.

$$= 1 - \frac{\overbrace{1}}{1{,}048{,}576} - \frac{\overbrace{\binom{20}{1}}}{1{,}048{,}576}$$

$$= \frac{1{,}048{,}576 - 1 - 20}{1{,}048{,}576} = \boxed{\frac{1{,}048{,}555}{1{,}048{,}576}}$$

$$\approx \boxed{0.99997997\ldots}$$

$$\approx \boxed{99.998\%}$$

(b)

$$P(\text{at most 3 heads}) = P(0 \text{ heads}) + P(\text{exactly 1 head}) + P(\text{exactly 2 heads}) + P(\text{exactly 3 heads})$$

$$= \frac{1}{2^{20}} + \frac{\overbrace{\binom{20}{1}}^{\text{pick 1 spot out of 20 for the head then fill the rest with tails in 1 way}}}{2^{20}} + \frac{\overbrace{\binom{20}{2}}^{\text{pick 2 spots out of 20 for the heads then fill the rest with tails in 1 way}}}{2^{20}} + \frac{\overbrace{\binom{20}{3}}^{\text{pick 3 spots out of 20 for the heads then fill the rest with tails in 1 way}}}{2^{20}}$$

$$= \frac{1+20+190+1140}{1{,}048{,}576} = \boxed{\frac{1{,}351}{1{,}048{,}576}}$$

$$\approx \boxed{0.00128841} \approx \boxed{0.1288\ \%}$$

(13) There are $6^4 = 1296$ ways to roll a 6-sided die four times in a row.

Let E be the event that a 3 occurs at least once in the four rolls. Then $\overline{E}$ is the event that no 3's occur in the four rolls.

1, 2, 4, 5, or 6	1, 2, 4, 5, or 6	1, 2, 4, 5, or 6	1, 2, 4, 5, or 6
5	5	5	5
possibilities	possibilities	possibilities	possibilities

← counting $\overline{E}$

$$5 \cdot 5 \cdot 5 \cdot 5 = 5^4 = 625$$

Thus, $P(\overline{E}) = \dfrac{625}{1296} \approx 0.48.$

So, $P(E) = 1 - P(\overline{E}) = 1 - \dfrac{625}{1296}$

$$= \frac{671}{1296} \approx \boxed{0.52}$$

14

The sample space has size

$$|S| = \binom{20}{5} = \frac{20!}{5!\,15!} = \frac{20 \cdot 19 \cdot 18 \cdot 17 \cdot 16 \cdot \cancel{15!}}{5!\,\cancel{15!}}$$

$$= \frac{1{,}860{,}480}{120} = 15{,}504$$

To count how many ways we can pick 5 numbers so the smallest number is larger than 6 we must pick 5 numbers from the 14 circled below.

1, 2, 3, 4, 5, 6, $\boxed{7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20}$

This can be done in $\binom{14}{5} = \frac{14!}{5!\,9!} = \frac{14 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot \cancel{9!}}{5!\,\cancel{9!}}$

$$= \frac{240{,}240}{120} = 2{,}002$$

Thus the probability is

$$\boxed{\frac{2{,}002}{15{,}504}} \approx \boxed{0.129} \approx \boxed{12.9\,\%}$$

15. Recall there are

$$\binom{47}{5} \cdot 27 = 41{,}416{,}353 \text{ possible tickets}$$

(a) The number of tickets that get 2 of the 5 lucky #s correct and the mega number is

$$\frac{\underbrace{\binom{5}{2}}_{\substack{\text{pick 2 of the}\\ \text{5 winning lucky}\\ \text{numbers}}} \cdot \underbrace{\binom{42}{3}}_{\substack{\text{pick 3 non-}\\ \text{winning lucky}\\ \text{numbers}}} \cdot \underbrace{\binom{1}{1}}_{\substack{\text{pick the}\\ \text{winning mega}\\ \text{number}}}}{41{,}416{,}353} = \frac{(10)(11{,}480)}{41{,}416{,}353} = \boxed{\frac{114{,}800}{41{,}416{,}353}}$$

$$\approx \boxed{0.00277\ldots}$$

$$\approx \boxed{0.277\%}$$

(b) The number of tickets that get 4 of the 5 lucky #s correct and the mega number is

$$\frac{\underbrace{\binom{5}{4}}_{\substack{\text{pick 4}\\ \text{of the}\\ \text{5 winning}\\ \text{lucky}\\ \text{numbers}}} \cdot \underbrace{\binom{42}{1}}_{\substack{\text{pick 1}\\ \text{non-}\\ \text{winning}\\ \text{luck}\\ \text{number}}} \cdot \underbrace{\binom{1}{1}}_{\substack{\text{pick}\\ \text{the}\\ \text{winning}\\ \text{mega}\\ \text{number}}}}{41,416,353} = \frac{(5)(42)}{41,416,353} = \boxed{\frac{210}{41,416,353}}$$

$$\approx \boxed{0.00000507\ldots}$$

$$\approx \boxed{0.000507\,\%}$$

16 There are 49 remaining cards. Thus, there are $\binom{49}{2} = \frac{49!}{2!\,47!} = \frac{49 \cdot 48 \cdot 47!}{2!\,47!} = \frac{49 \cdot 48}{2} = 1{,}176$ possible two card combinations that you can get.

(a) There are $13 - 3 = 10$ remaining clubs. So, the odds of getting two clubs is

$$\frac{\binom{10}{2}}{\binom{49}{2}} = \boxed{\frac{45}{1{,}176}} \approx \boxed{0.038\ldots} \approx \boxed{3.8\ \%}$$

(b) The cards that give you a straight are

$$\underbrace{A^{?}\ \ 5^{?}}_{\text{gives}} \qquad \text{or} \qquad \underbrace{5^{?}\ \ 6^{?}}_{\text{gives}}$$

$$A^{?}\ \ 2^{\clubsuit}\ \ 3^{\clubsuit}\ \ 4^{\clubsuit}\ \ 5^{?} \qquad\qquad 2^{\clubsuit}\ \ 3^{\clubsuit}\ \ 4^{\clubsuit}\ \ 5^{?}\ \ 6^{?}$$

where ? is any suit except you don't want to count $A^{\clubsuit}\ \ 5^{\clubsuit}$ or $5^{\clubsuit}\ \ 6^{\clubsuit}$ since those would give you a straight flush.

Thus, the number of hands that give you a straight but not a straight flush is

$$\underbrace{4 \cdot 4}_{\boxed{A^{?}}\,\boxed{5^{?}}} + \underbrace{4 \cdot 4}_{\boxed{5^{?}}\,\boxed{6^{?}}} - \underbrace{2}_{\substack{\boxed{A^{\clubsuit}}\,\boxed{5^{\clubsuit}} \\ \boxed{5^{\clubsuit}}\,\boxed{6^{\clubsuit}}}} = 30$$

So, the probability is $\boxed{\dfrac{30}{1,176}} \approx \boxed{0,02551} \approx \boxed{2.551\,\%}$

(c) The cards that give you a straight flush are $\boxed{A^{\clubsuit}}\,\boxed{5^{\clubsuit}}$ and $\boxed{5^{\clubsuit}}\,\boxed{6^{\clubsuit}}$. Thus, the probability is $\boxed{\dfrac{2}{1,176}} \approx \boxed{0.0017\ldots} \approx \boxed{0.17\,\%}$

17

There are $\binom{52}{2} = \dfrac{52!}{2!\,50!} = \dfrac{52 \cdot 51}{2} = 1326$ ways to be dealt two cards.

(a) There are four aces: $A\spadesuit$, $A\clubsuit$, $A\heartsuit$, $A\diamondsuit$

Thus there are $\binom{4}{2}$ possible ways to be dealt two aces.

$$\binom{4}{2} = \frac{4!}{2!\,2!} = \frac{4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1 \cdot 2 \cdot 1} = 6$$

Thus the probability of such an event is

$$\frac{\binom{4}{2}}{\binom{52}{2}} = \frac{6}{1326} \approx \boxed{0.004524890\ldots}$$

$$\boxed{\text{or } \approx 0.45\%}$$

(b)

There are 13 possible face values:

A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K

Each face value has 4 suits.

Thus, there are

$$\underbrace{13}_{\text{choose the face value}} \cdot \underbrace{\binom{4}{2}}_{\text{choose two of the 4 cards, ie choose two from } \heartsuit, \diamondsuit, \spadesuit, \clubsuit} = 13 \cdot \frac{4!}{2!2!} = 13 \cdot 6 = 78$$

ways to get two cards of the same face value.

Thus, the probability of such an event is

$$\frac{13 \cdot \binom{4}{2}}{\binom{52}{2}} = \frac{78}{1326} \approx \boxed{0.5882\ldots}$$

$$\boxed{\text{or } \approx 5.88\%}$$

(c) How many blackjacks are there?

There are 4 aces:

A♥, A♦, A♣, A♠

There are 16 tens, jacks, queens, kings:

10♥, 10♦, 10♣, 10♠

J♥, J♦, J♣, J♠

Q♥, Q♦, Q♣, Q♠

K♥, K♦, K♣, K♠

There are $\underbrace{\binom{4}{1}}_{\text{choose an ace}} \cdot \underbrace{\binom{16}{1}}_{\text{choose a 10, J, Q, or K}} = 4 \cdot 16 = 64$ blackjacks

Thus, the probability of being dealt a blackjack is $\dfrac{64}{\binom{52}{2}} = \boxed{\dfrac{64}{1326}} \approx \boxed{0.048\ldots}$

$\approx \boxed{4.8\%}$

(18) Recall from class that there are

$$\binom{52}{5} = 2,598,960$$

possible 5-card poker hands.

(a) We need to count the number of flushes.

Step 1: Pick the suit.

♠, ♣, ♥, ♦ ← $\binom{4}{1} = 4$ possibilites

say we pick this one

?♥ ?♥ ?♥ ?♥ ?♥ ← after step 1

Step 2: Pick 5 face values

A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K ← $\binom{13}{5} = \frac{13!}{5!8!} = \frac{13\cdot 12\cdot 11\cdot 10\cdot 9\cdot \cancel{8!}}{120\cdot \cancel{8!}} = 1287$ possibilities

say we picked these 5

2♥ 5♥ 6♥ 9♥ 10♥ ← after step 2

So, # flushes is $4 \cdot 1287 = 5148$.

Thus, the probability of getting a flush is

$$\frac{5148}{2{,}598{,}960} \approx \boxed{0.00198\ldots} \approx \boxed{0.198\,\%}$$

(b) Let's count the # of three of a kinds.

Step 1: Pick the face value for the three of a kind.

A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K ← $\boxed{\binom{13}{1} = 13}$

say we picked 5

$\boxed{5\ ?}$ $\boxed{5\ ?}$ $\boxed{5\ ?}$ $\boxed{?\ ?}$ $\boxed{?\ ?}$ ← after step 1

Step 2: Pick the suits for the three of a kind.

♥, ♦, ♠, ♣

say we picked these three

$\binom{4}{3} = \frac{4!}{3!\,1!} = 4$

5♥	5♠	5♣	? ?	? ?

after step 2

Step 3: Pick the face values for the non-three of a kind part

A, 2, 3, 4, ~~5~~, 6, 7, 8, 9, 10, J, Q, K

cant pick

say we picked these

$\binom{12}{2} = 66$

5♥	5♠	5♣	2 ?	8 ?

after step 3

Step 4: Pick the suits for the non-three of a kind part

♡, ♢, ♠, ♣ ← $\binom{4}{1} = 4$

♡, ♢, ♠, ♣ ← $\binom{4}{1} = 4$

(say we picked the circled above)

5♡ 5♠ 5♣ 2♢ 8♡ ← after step 4

Combining all 4 steps gives

$$13 \cdot 4 \cdot 66 \cdot 4 \cdot 4 = 54{,}912$$

three of a kinds.

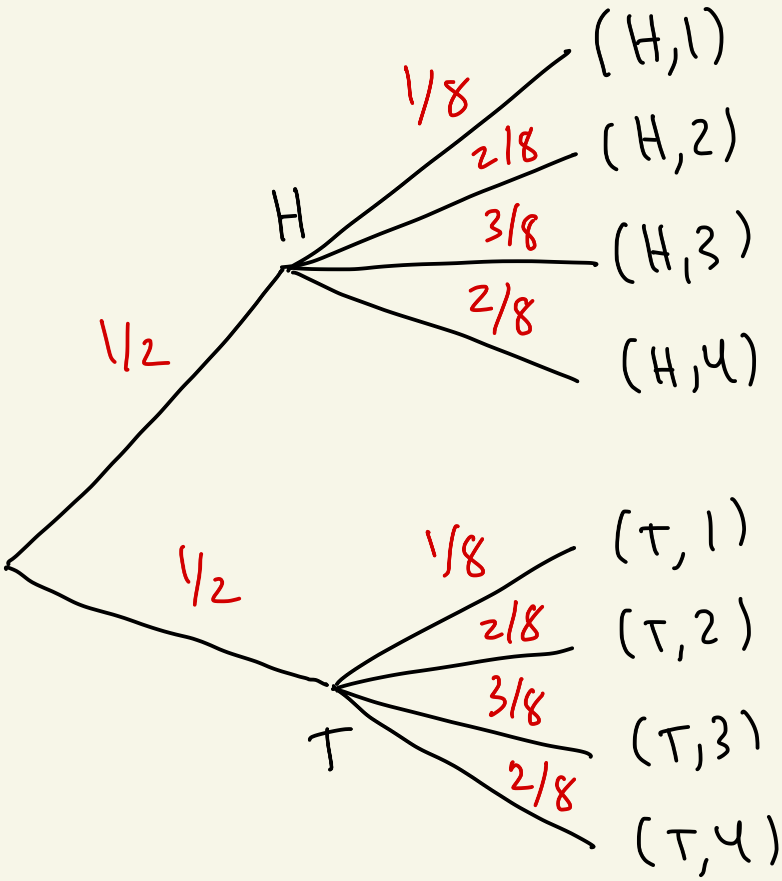
Thus, the probability of getting a three of a kind is

$$\frac{54{,}912}{2{,}598{,}960} \approx \boxed{0.0211\ldots} \approx \boxed{2.11\ \%}$$

19

$$S = \{(H,1), (H,2), (H,3), (H,4), (T,1), (T,2), (T,3), (T,4)\}$$

Ω is the set of all subsets of S



$P(H,1) = \frac{1}{2} \cdot \frac{1}{8} = \frac{1}{16}$ $\qquad$ $P(T,1) = \frac{1}{2} \cdot \frac{1}{8} = \frac{1}{16}$

$P(H,2) = \frac{1}{2} \cdot \frac{2}{8} = \frac{2}{16}$ $\qquad$ $P(T,2) = \frac{1}{2} \cdot \frac{2}{8} = \frac{2}{16}$

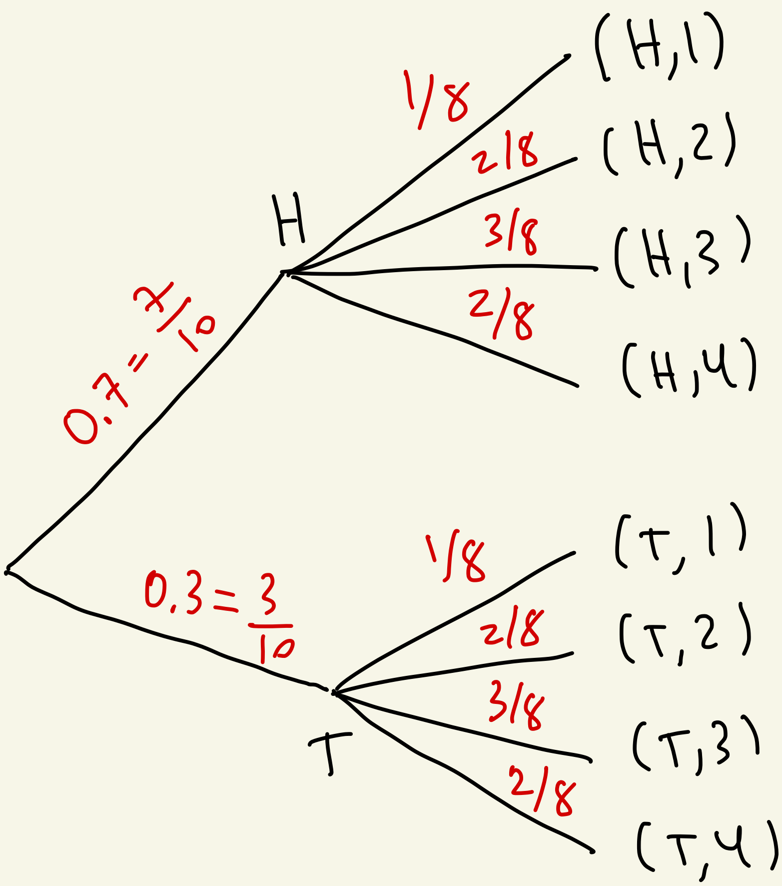
$P(H,3) = \frac{1}{2} \cdot \frac{3}{8} = \frac{3}{16}$ $\qquad$ $P(T,3) = \frac{1}{2} \cdot \frac{3}{8} = \frac{3}{16}$

$P(H,4) = \frac{1}{2} \cdot \frac{2}{8} = \frac{2}{16}$ $\qquad$ $P(T,4) = \frac{1}{2} \cdot \frac{2}{8} = \frac{2}{16}$

(20)

$$S = \{ (H,1), (H,2), (H,3), (H,4), (T,1), (T,2), (T,3), (T,4) \}$$

Ω is the set of all subsets of S



$P(H,1) = \frac{7}{10} \cdot \frac{1}{8} = \frac{7}{80}$ $\qquad$ $P(T,1) = \frac{3}{10} \cdot \frac{1}{8} = \frac{3}{80}$

$P(H,2) = \frac{7}{10} \cdot \frac{2}{8} = \frac{14}{80}$ $\qquad$ $P(T,2) = \frac{3}{10} \cdot \frac{2}{8} = \frac{6}{80}$

$P(H,3) = \frac{7}{10} \cdot \frac{3}{8} = \frac{21}{80}$ $\qquad$ $P(T,3) = \frac{3}{10} \cdot \frac{3}{8} = \frac{9}{80}$

$P(H,4) = \frac{7}{10} \cdot \frac{2}{8} = \frac{14}{80}$ $\qquad$ $P(T,4) = \frac{3}{10} \cdot \frac{2}{8} = \frac{6}{80}$