Math 4740 Homework 5 Solutions

① $n = 15$ flips

$p = \frac{1}{2}$ (probability of getting heads)

(a) $E[X] = np = \frac{15}{2} = \boxed{7.5}$

(b) $P(X=3) = \binom{15}{3} \cdot \left(\frac{1}{2}\right)^3 \cdot \left(1-\frac{1}{2}\right)^{15-3}$

$= \frac{15!}{3!\,12!} \cdot \frac{1}{2^3} \cdot \frac{1}{2^{12}} = \frac{15 \cdot 14 \cdot 13 \cdot \cancel{12!}}{6 \cdot \cancel{12!}} \cdot \frac{1}{2^{15}}$

$= \boxed{\frac{455}{32{,}768}} \approx \boxed{0.0138855\ldots}$

(c) $P(X \le 2) = P(X=0) + P(X=1) + P(X=2)$

$= \binom{15}{0} \cdot \left(\frac{1}{2}\right)^0 \underbrace{\left(\frac{1}{2}\right)^{15}}_{\left(1-\frac{1}{2}\right)^{15}} + \binom{15}{1} \cdot \left(\frac{1}{2}\right)^1 \cdot \underbrace{\left(\frac{1}{2}\right)^{14}}_{\left(1-\frac{1}{2}\right)^{14}} + \binom{15}{2} \cdot \left(\frac{1}{2}\right)^2 \cdot \underbrace{\left(\frac{1}{2}\right)^{13}}_{\left(1-\frac{1}{2}\right)^{13}}$

$= 1 \cdot \frac{1}{2^{15}} + 15 \cdot \frac{1}{2^{15}} + 105 \cdot \frac{1}{2^{15}}$

$= \boxed{\frac{121}{32{,}768}} \approx \boxed{0.00369263\ldots}$

(d) We want $P(X \geq 2)$.

This would involve calculating

$$P(X \geq 2) = P(X=2) + P(X=3) + \dots + P(X=15)$$

It's easier to calculate

$$P(X \geq 2) = 1 - P(X < 2)$$

$$= 1 - P(X=0) - P(X=1)$$

$$= 1 - \binom{15}{0} \cdot \left(\frac{1}{2}\right)^0 \cdot \left(\frac{1}{2}\right)^{15} - \binom{15}{1} \cdot \left(\frac{1}{2}\right)^1 \cdot \left(\frac{1}{2}\right)^{14}$$

$$= 1 - 1 \cdot \frac{1}{2^{15}} - 15 \cdot \frac{1}{2^{15}}$$

$$= 1 - \frac{16}{2^{15}}$$

$$= \boxed{\frac{32{,}752}{32{,}768}} \approx \boxed{0.999512}$$

② $n = 10$ rolls of the dice

$$p = \underbrace{\frac{6}{36}}_{\substack{\text{probability}\\\text{of rolling}\\\text{a 7}}} + \underbrace{\frac{2}{36}}_{\substack{\text{probability}\\\text{of rolling}\\\text{an 11}}} = \frac{8}{36} = \frac{2}{9}$$ ← probability of rolling 7 or 11.

(a) $E[X] = np = 10 \cdot \left(\frac{2}{9}\right) = \boxed{\frac{20}{9}} \approx \boxed{2.22}$

(b) $P(X=5) = \binom{10}{5} \cdot \left(\frac{2}{9}\right)^5 \cdot \left(1 - \frac{2}{9}\right)^{10-5}$

$$= \frac{10!}{5!5!} \cdot \frac{2^5}{9^5} \cdot \frac{7^5}{9^5} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5!}{120 \cdot 5!} \cdot \frac{2^5 \cdot 7^5}{9^{10}}$$

$$= 252 \cdot \frac{32 \cdot 16{,}807}{3{,}486{,}784{,}401}$$

$$= \boxed{\frac{135{,}531{,}648}{3{,}486{,}784{,}401}} \approx \boxed{0.03887\ldots}$$

③ $n = 10$

$p = \frac{1}{36}$ ← (probability of rolling double sixes)

$1-p = \frac{35}{36}$

$X = \#$ of successes

$$P(X \geq 3) = P(X=3) + P(X=4) + \dots + P(X=10)$$

Instead we do this:

$$P(X \geq 3) = 1 - P(X<3)$$

$$= 1 - P(X=0) - P(X=1) - P(X=2)$$

$$= 1 - \binom{10}{0}\cdot\left(\frac{1}{36}\right)^0\left(\frac{35}{36}\right)^{10} - \binom{10}{1}\cdot\left(\frac{1}{36}\right)^1\cdot\left(\frac{35}{36}\right)^9$$

$$- \binom{10}{2}\cdot\left(\frac{1}{36}\right)^2\cdot\left(\frac{35}{36}\right)^8$$

$$= 1 - \frac{35^{10}}{36^{10}} - 10\cdot\frac{35^9}{36^{10}} - 45\cdot\frac{35^8}{36^{10}}$$

$$= \boxed{\frac{11{,}259{,}376{,}953{,}125}{11{,}284{,}439{,}629{,}824}}$$

$$\approx \boxed{0.997779\dots}$$

④

(a) $$\frac{\overbrace{\binom{4}{1}}^{\text{choose an ace}} \cdot \overbrace{\binom{16}{1}}^{\text{choose a 10, J, Q, or K}}}{\binom{52}{2}} = \frac{4 \cdot 16}{\left(\frac{52 \cdot 51}{2}\right)} = \frac{64}{1326} = \boxed{\frac{32}{663}} \approx \boxed{0.04826}$$

there are $16 = 4+4+4+4$ 10, J, Q, K's

(b) Let X be the number of blackjacks in $n = 20$ experiments. We have $p = \frac{32}{663} \approx 0.048$ is the probability of success and $1 - p = \frac{631}{663} \approx 0.952$

Then,

$$P(X \geq 2) = 1 - P(X < 2)$$
$$= 1 - P(X = 0) - P(X = 1)$$
$$= 1 - \binom{20}{0} \cdot \left(\frac{32}{663}\right)^0 \cdot \left(\frac{631}{663}\right)^{20} - \binom{20}{1} \cdot \left(\frac{32}{663}\right)^1 \cdot \left(\frac{631}{663}\right)^{19}$$
$$\approx 1 - 1 \cdot 1 \cdot (0.952)^{20} - 20 \cdot (0.048)(0.952)^{19}$$
$$\approx 1 - 0.3738 - 0.3769$$
$$\approx \boxed{0.2493}$$

⑤ There are 18 black numbers on the Roulette wheel out of 38 total. Thus,

$$n = 5, \quad p = \frac{18}{38} = \frac{9}{19}, \quad 1-p = \frac{10}{19}$$

(a) / (b)

$$p(0) = P(X=0) = \binom{5}{0} \cdot \left(\frac{9}{19}\right)^0 \cdot \left(\frac{10}{19}\right)^5 = \frac{100{,}000}{2{,}476{,}099} \approx 0.04\ldots$$

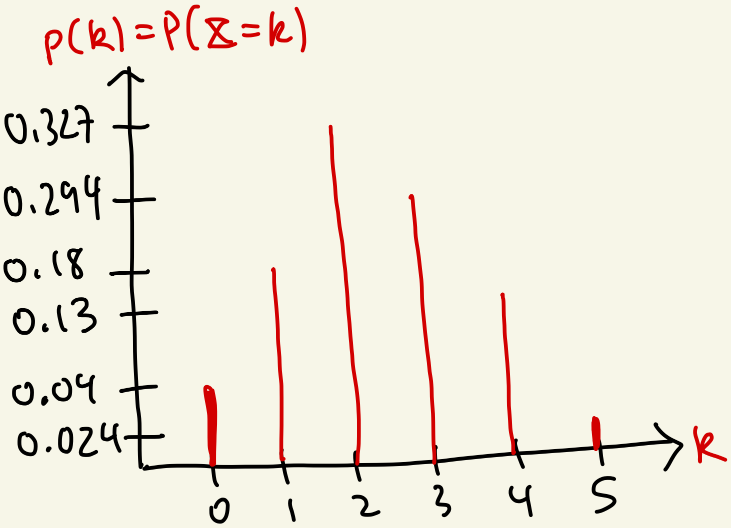
$$p(1) = P(X=1) = \binom{5}{1} \cdot \left(\frac{9}{19}\right)^1 \cdot \left(\frac{10}{19}\right)^4 = \frac{450{,}000}{2{,}476{,}099} \approx 0.18\ldots$$

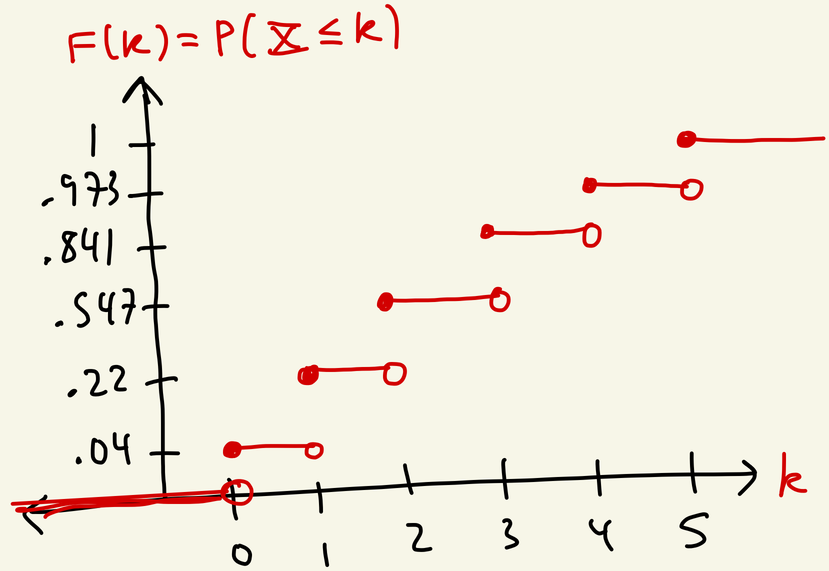
$$p(2) = P(X=2) = \binom{5}{2}\left(\frac{9}{19}\right)^2\left(\frac{10}{19}\right)^3 = \frac{810{,}000}{2{,}476{,}099} \approx 0{,}327\ldots$$

$$p(3) = P(X=3) = \binom{5}{3}\left(\frac{9}{19}\right)^3\left(\frac{10}{19}\right)^2 = \frac{729{,}000}{2{,}476{,}099} \approx 0.294\ldots$$

$$p(4) = P(X=4) = \binom{5}{4}\left(\frac{9}{19}\right)^4\left(\frac{10}{19}\right)^1 = \frac{328{,}050}{2{,}476{,}099} \approx 0.132\ldots$$

$$p(5) = P(X=5) = \binom{5}{5}\left(\frac{9}{19}\right)^5\left(\frac{10}{19}\right)^0 = \frac{59{,}049}{2{,}476{,}099} \approx 0.024\ldots$$





$\boxed{\text{(c)}}$

$$P(X \geq 3) = P(X=3) + P(X=4) + P(X=5)$$

$$= \boxed{\frac{1{,}116{,}099}{2{,}476{,}099}} \approx \boxed{0.450749\ldots}$$

$\boxed{\text{(d)}}$ $$E[X] = n \cdot p = 5 \cdot \frac{9}{19} = \boxed{\frac{45}{19}}$$