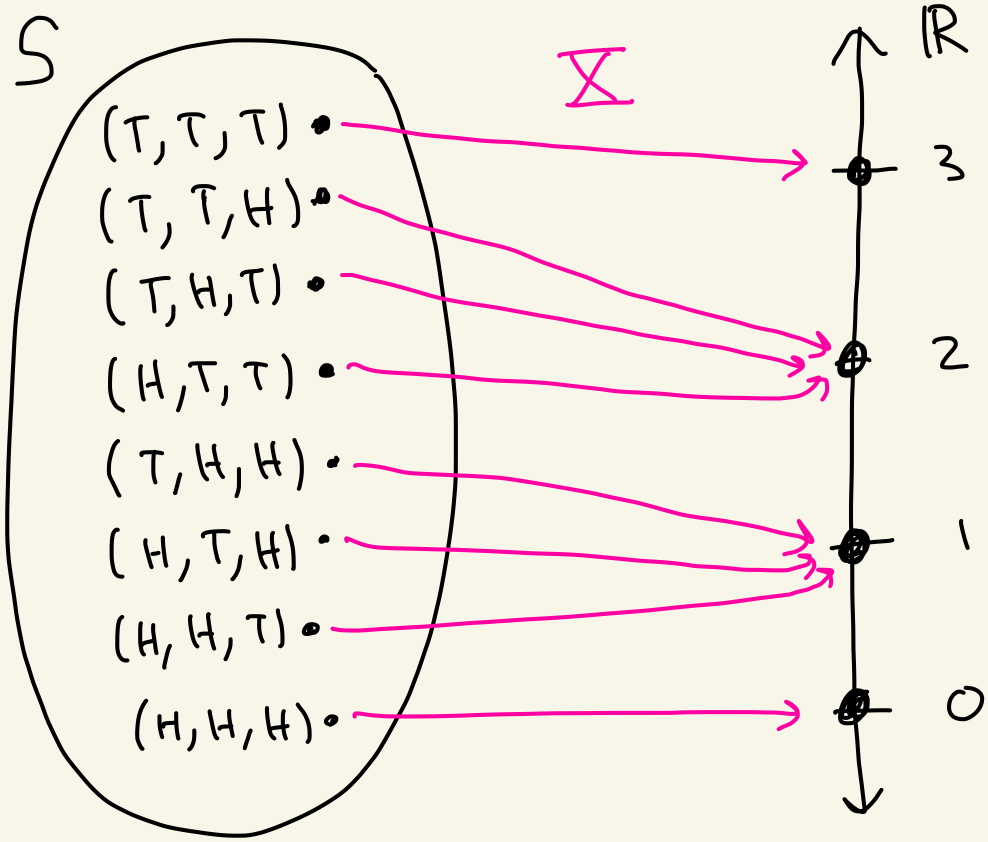
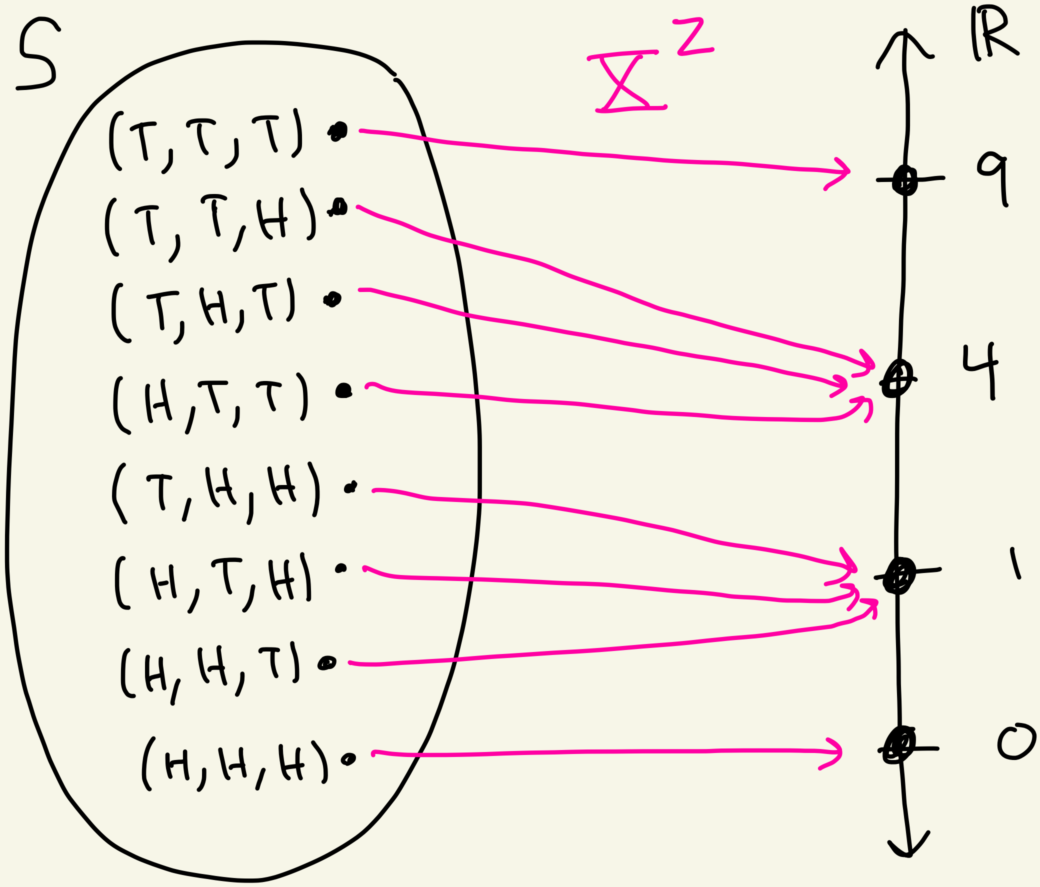
# Math 4740
# Homework 6
# Solutions

1 (a)


S
X
ℝ
(T,T,T)
(T,T,H)
(T,H,T)
(H,T,T)
(T,H,H)
(H,T,H)
(H,H,T)
(H,H,H)
3
2
1
0

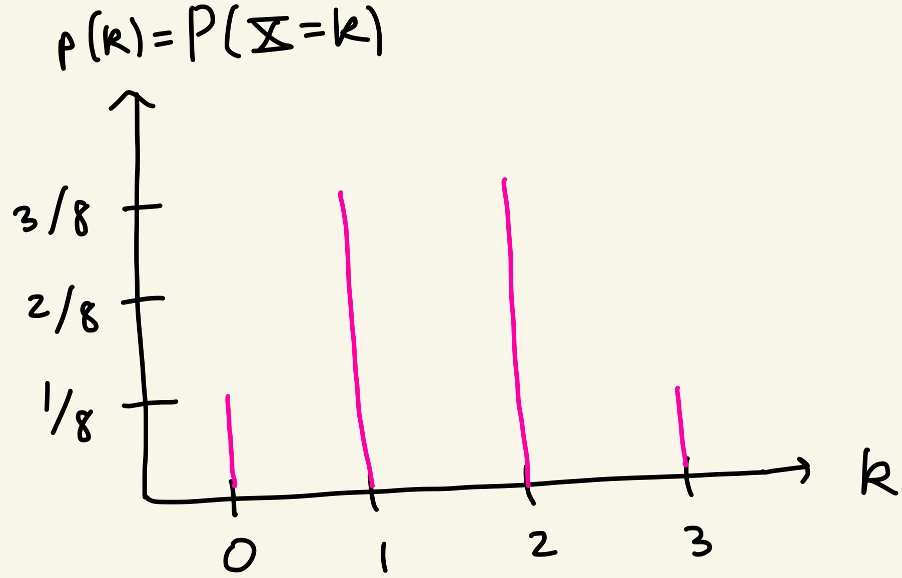

S
X²
ℝ
(T,T,T)
(T,T,H)
(T,H,T)
(H,T,T)
(T,H,H)
(H,T,H)
(H,H,T)
(H,H,H)
9
4
1
0

$P(X=0) = \frac{1}{8}$

$P(X=1) = \frac{3}{8}$

$P(X=2) = \frac{3}{8}$

$P(X=3) = \frac{1}{8}$




① (b)

$$E[X] = (0)\left(\frac{1}{8}\right) + (1)\left(\frac{3}{8}\right) + (2)\left(\frac{3}{8}\right) + (3)\left(\frac{1}{8}\right)$$

$$= \frac{3+6+3}{8} = \frac{12}{8} = \boxed{\frac{3}{2}}$$

$$E[X^2] = (0^2)\left(\frac{1}{8}\right) + (1^2)\left(\frac{3}{8}\right) + (2^2)\left(\frac{3}{8}\right) + (3^2)\left(\frac{1}{8}\right)$$

$$= \frac{3+12+9}{8} = \frac{24}{8} = 3$$

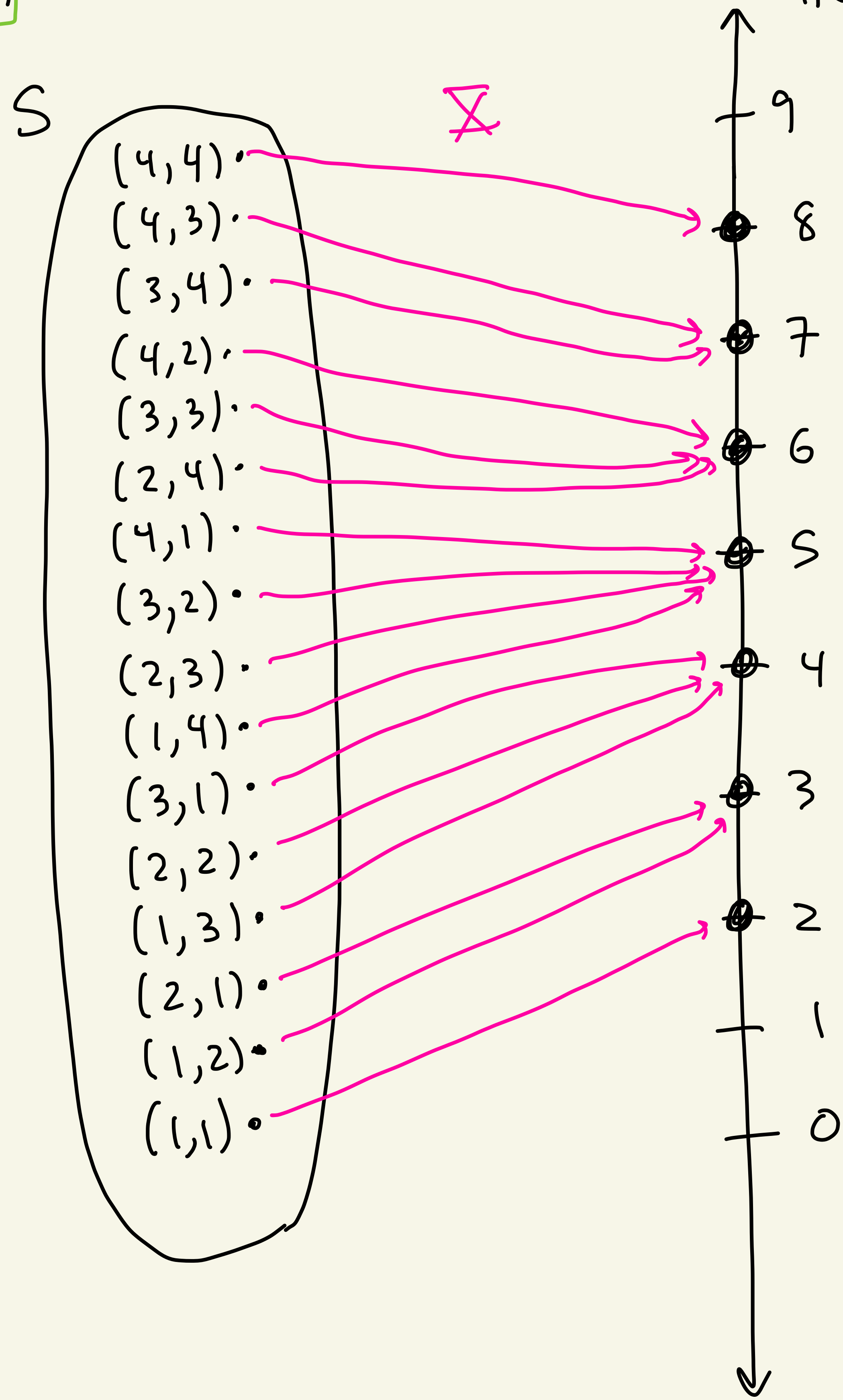
$$\mathrm{Var}(X) = E[X^2] - \left(E[X]\right)^2 = 3 - \left(\frac{3}{2}\right)^2$$

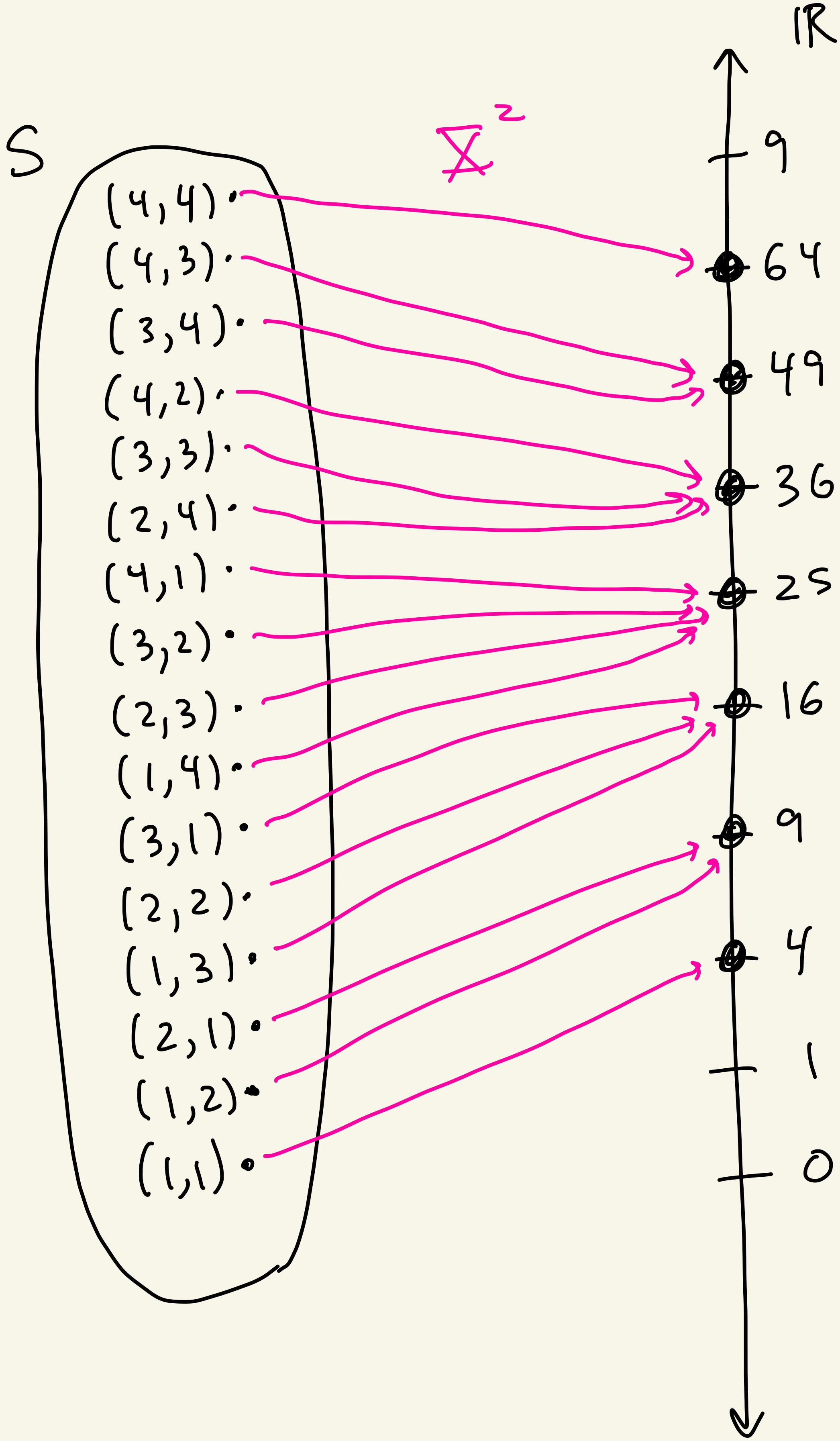
$$= 3 - \frac{9}{4}$$

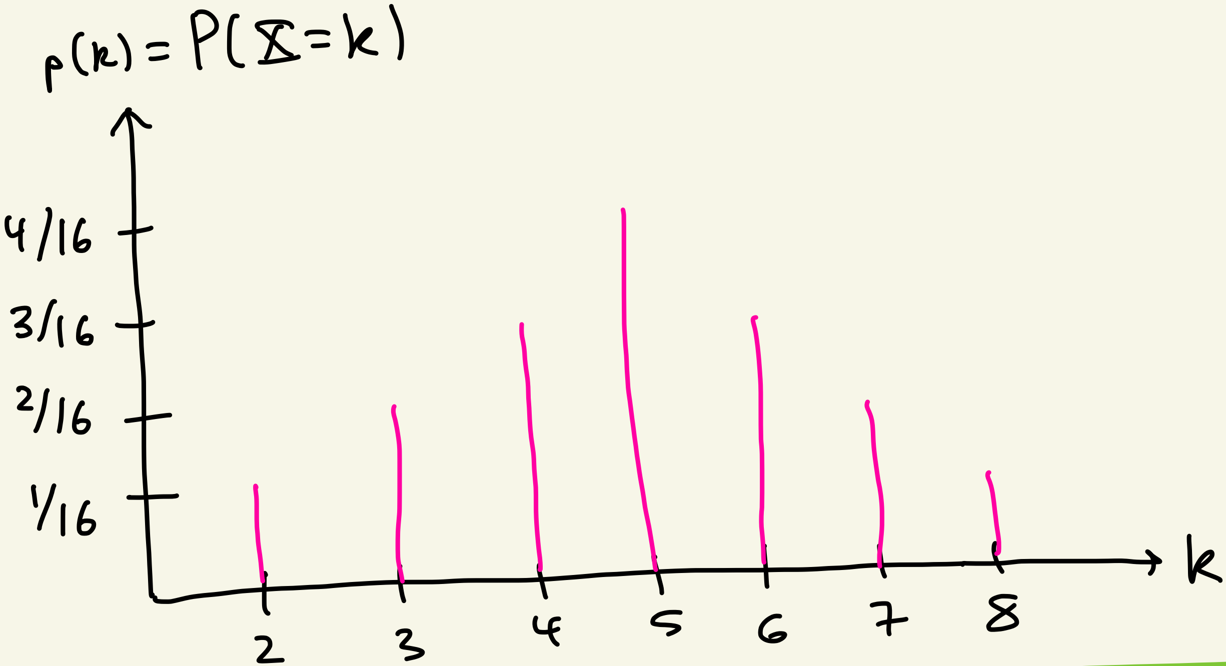
$$= \frac{12-9}{4} = \boxed{\frac{3}{4}}$$

$$\sigma = \sqrt{\mathrm{Var}(X)} = \sqrt{\frac{3}{4}} \approx \boxed{0.866}$$

② (a)




S
$X^2$
ℝ
(4,4)
(4,3)
(3,4)
(4,2)
(3,3)
(2,4)
(4,1)
(3,2)
(2,3)
(1,4)
(3,1)
(2,2)
(1,3)
(2,1)
(1,2)
(1,1)
9
64
49
36
25
16
9
4
1
0




② (b)

$$E[X] = (2)\left(\tfrac{1}{16}\right) + (3)\left(\tfrac{2}{16}\right) + (4)\left(\tfrac{3}{16}\right) + (5)\left(\tfrac{4}{16}\right) + (6)\left(\tfrac{3}{16}\right) + (7)\left(\tfrac{2}{16}\right) + (8)\left(\tfrac{1}{16}\right)$$

$$= \frac{2+6+12+20+18+14+8}{16} = \frac{80}{16} = \boxed{5}$$

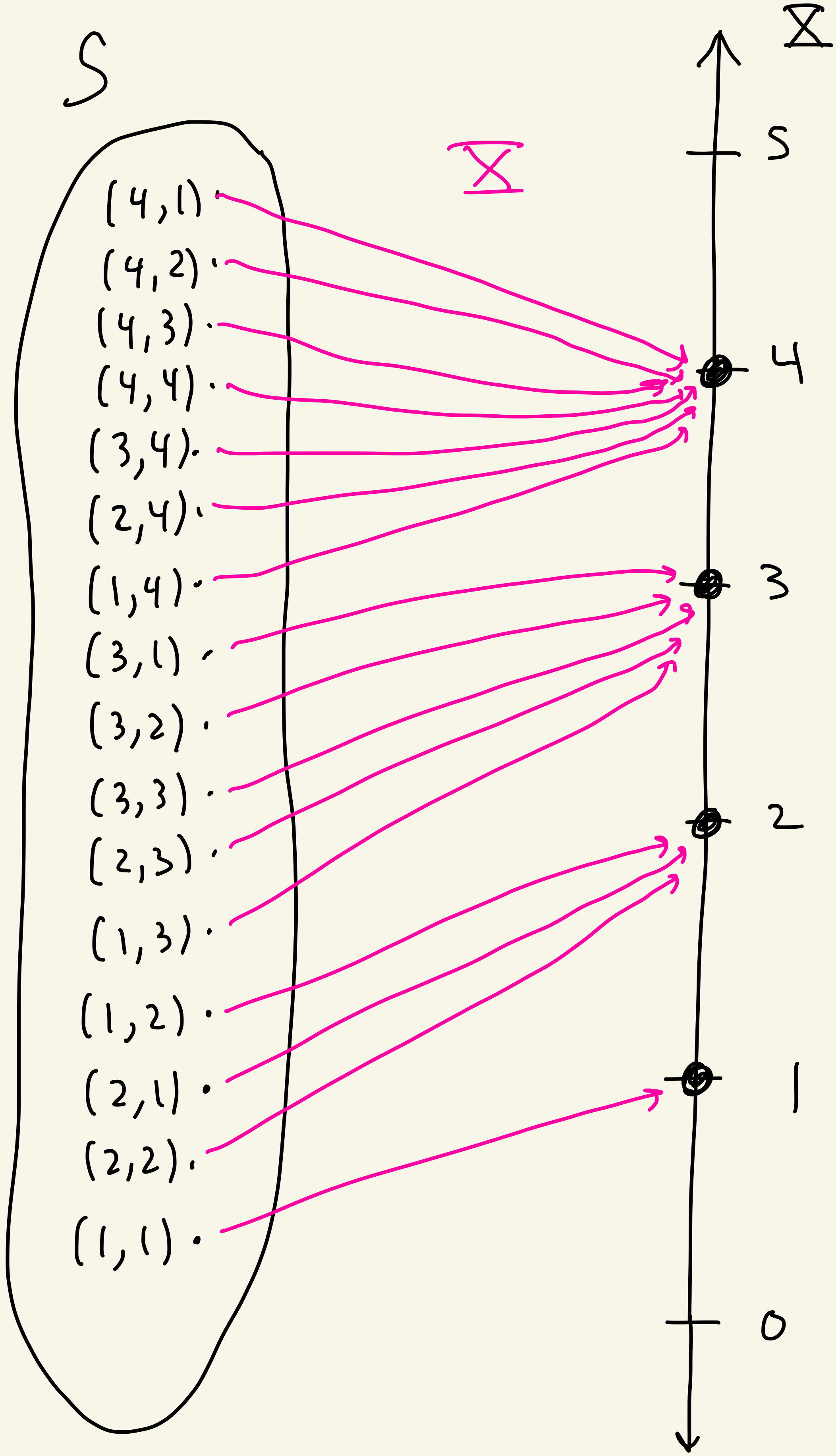
$$E[X^2] = (2^2)\left(\tfrac{1}{16}\right) + (3^2)\left(\tfrac{2}{16}\right) + (4^2)\left(\tfrac{3}{16}\right) + (5^2)\left(\tfrac{4}{16}\right) + (6^2)\left(\tfrac{3}{16}\right) + (7^2)\left(\tfrac{2}{16}\right) + (8^2)\left(\tfrac{1}{16}\right)$$

$$= \frac{4+18+48+100+108+98+64}{16} = \frac{440}{16} = \boxed{27.5}$$

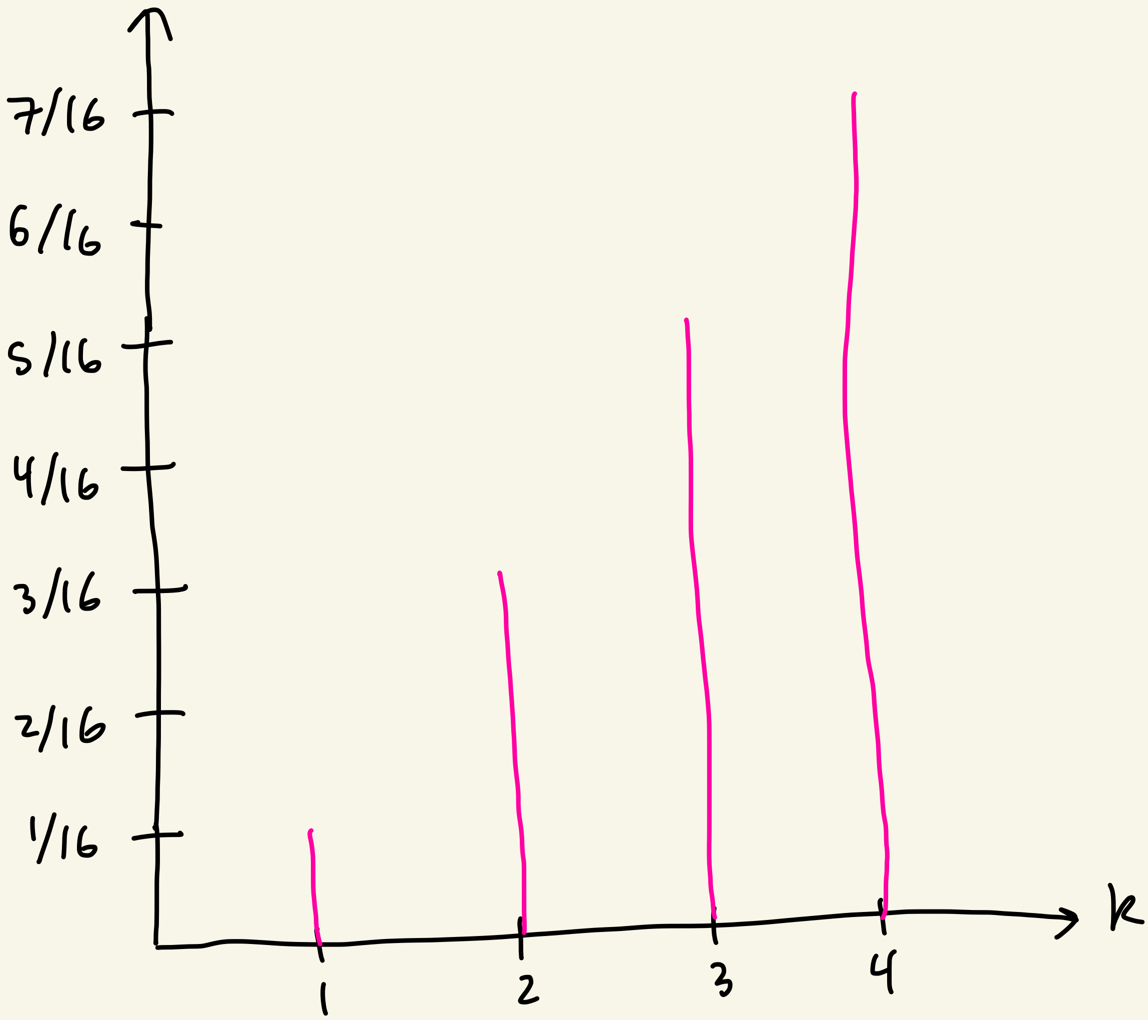
$$Var(X) = E[X^2] - (E[X])^2 = 27.5 - 5^2 = \boxed{2.5}$$

$$\sigma = \sqrt{Var(X)} = \sqrt{2.5} \approx \boxed{1.58}$$

3 (a)



$p(k) = P(X = k)$




3(b) $$E[X] = (1)\left(\frac{1}{16}\right) + (2)\left(\frac{3}{16}\right) + (3)\left(\frac{5}{16}\right) + (4)\left(\frac{7}{16}\right)$$

$$= \frac{1 + 6 + 15 + 28}{16} = \boxed{\frac{50}{16}}$$

$$E[X^2] = (1^2)\left(\frac{1}{16}\right) + (2^2)\left(\frac{3}{16}\right) + (3^2)\left(\frac{5}{16}\right) + (4^2)\left(\frac{7}{16}\right)$$

$$= \frac{1 + 12 + 45 + 112}{16} = \frac{170}{16}$$

$$\mathrm{Var}(X) = E[X^2] - (E[X])^2 = \frac{170}{16} - \left(\frac{50}{16}\right)^2$$

$$= \frac{2720}{256} - \frac{2500}{256}$$

$$= \frac{220}{256} = \frac{55}{64} \approx \boxed{0.859}$$

$$\sigma = \sqrt{\mathrm{Var}(X)} = \sqrt{\frac{55}{64}} \approx \boxed{0.927}$$

4 (a)

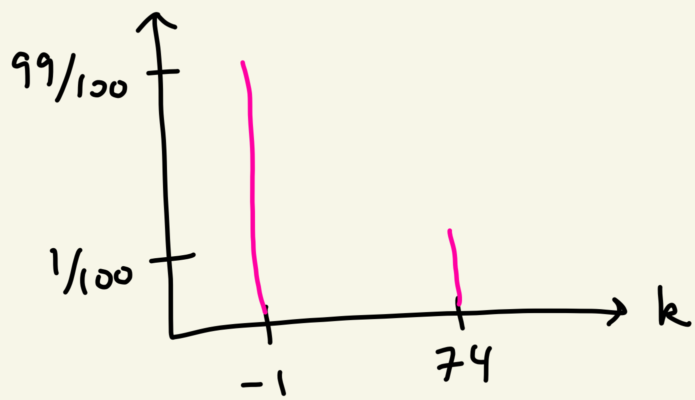
Let $X_A$ be the expected value of game A and $X_B$ be the expected value of game B.

## GAME A

$$P(X_A = -1) = \frac{99}{100}$$

$$P(X_A = 74) = \frac{1}{100}$$

$p(k) = P(X_A = k)$

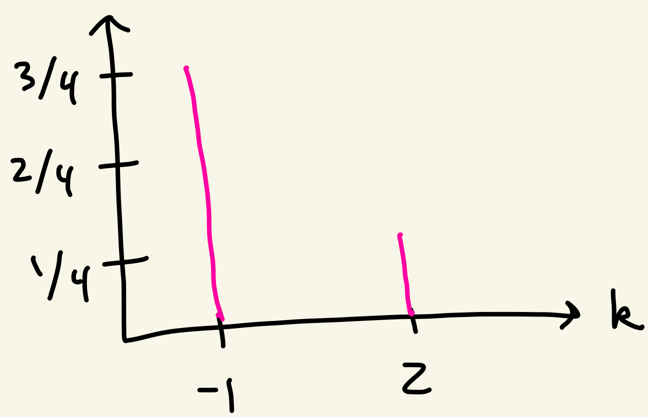



## GAME B

$$P(X_B = -1) = 3/4$$

$$P(X = 2) = 1/4$$

$p(k) = P(X_B = k)$



## 4(b)

$$E[X_A] = (-1)\left(\frac{99}{100}\right) + (74)\left(\frac{1}{100}\right) = \frac{-99+74}{100} = \frac{-25}{100} = \boxed{-\frac{1}{4}}$$

$$E[X_A^2] = (-1)^2\left(\frac{99}{100}\right) + (74^2)\left(\frac{1}{100}\right) = \frac{99+5476}{100} = \frac{5575}{100}$$

$$\text{Var}(X_A) = E[X_A^2] - \left(E[X_A]\right)^2 = \frac{5575}{100} - \left(-\frac{1}{4}\right)^2$$

$$= 55.75 - 0.0625 = \boxed{55.6875}$$

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## 4(c)

$$E[X_B] = (-1)\left(\frac{3}{4}\right) + (2)\left(\frac{1}{4}\right) = \boxed{-\frac{1}{4}}$$

$$E[X_B^2] = (-1)^2\left(\frac{3}{4}\right) + (2^2)\left(\frac{1}{4}\right) = \frac{3+4}{4} = \frac{7}{4}$$

$$\text{Var}(X) = E[X_B^2] - \left(E[X_B]\right)^2 = \frac{7}{4} - \left(-\frac{1}{4}\right)^2$$

$$= \frac{7}{4} - \frac{1}{16} = \frac{28-1}{16} = \frac{27}{16} \approx \boxed{1.6875}$$

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**4(d)** Both games have expected value $-0.25$ cents per attempt in the long run. However game A has huge swings compared to game B which is reflected in the variance. It's up to the player to decide based on their risk tolerance.

5

(a) Let $t = k\sigma$ in Chebyshev's inequality.

(b) Let $k = 2$ in part (a).

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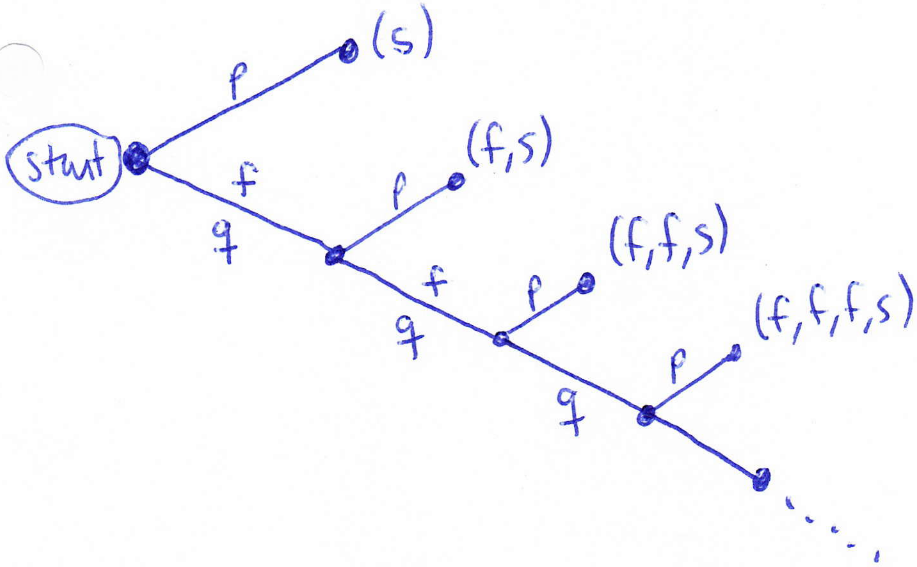
6 Let s represent success and
f represent failure.

(a) $S = \{ (s), (f,s), (f,f,s), (f,f,f,s), \ldots \}$

It's like this tree:




$P((s)) = p$

$P((f,s)) = qp$

$P((f,f,s)) = q^2 p$

$\vdots$

$P((\underbrace{f,f,\ldots,f}_{n\ f\text{'s}},s)) = q^n p$

Let $X$ = # trials before first success occurs. Then,

$P(X=1) = p$

$P(X=2) = qp$

$P(X=3) = q^2 p$

$\vdots$

$P(X=k) = q^{k-1} p$ where $k \geq 1$.

(a) continued... Define $P$ so that $P(E) = \sum_{\omega \in E} P(\{\omega\})$, and the values $P(\{\omega\})$ are as on the previous page.

Note that this is a probability space since

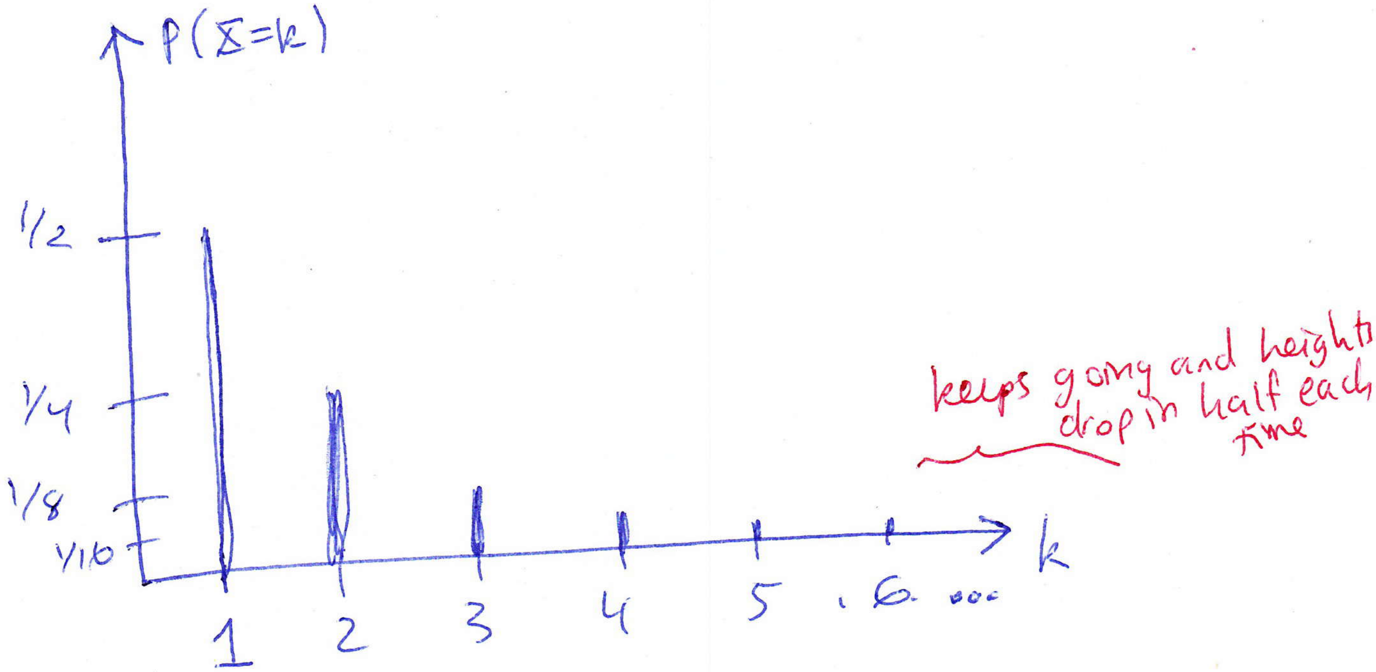
$$P(S) = \sum_{\omega \in S} P(\omega) = p + qp + q^2 p + q^3 p + \dots$$

$$= p\left[1 + q + q^2 + q^3 + \dots\right]$$

$$= p\left[\frac{1}{1-q}\right] = \frac{p}{1-q} = \frac{p}{p} = 1.$$

~~So everything is good.~~

(b) $p = \frac{1}{2}$ , $q = \frac{1}{2}$ , $P(X=k) = \left(\frac{1}{2}\right)^{k-1}\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^{k}$




(c) Recall that $P(X=k) = pq^{k-1}$

$$E[X] = (1)(p) + (2)(pq) + (3)(pq^2) + (4)(pq^3) + \cdots$$

Suppose that $|x| < 1$. Then

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots$$

Differentiating both sides gives

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$$

Set $x = q$ which satisfies $|q| < 1$.

This gives

$$\frac{1}{p^2} = \frac{1}{(1-q)^2} = 1 + 2q + 3q^2 + 4q^3 + \cdots$$

Now multiply by $p$ to get

$$\frac{1}{p} = p + 2pq + 3pq^2 + 4pq^3 + \cdots = E[X].$$

Now let's compute $Var[X]$.

We need $E[X^2] = (1)^2 p + (2)^2 pq + (3)^2 pq^2 + (4)^2 pq^3 + \cdots$

From above, if $|x|<1$ then

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$$

So,

$$\frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + 4x^4 + \cdots$$

Differentiating gives

$$\frac{(1-x)^2 - x[2(1-x)(-1)]}{(1-x)^4} = 1 + (2^2)x + (3^2)x^2 + (4^2)x^3 + \cdots$$

So,

$$\frac{1-x^2}{(1-x)^4} = (1)^2 + (2)^2 x + (3)^2 x^2 + (4)^2 x^3 + \cdots$$

Set $q = x$ and multiply through by $p$ to get

$$\frac{p(1-q^2)}{(1-q)^4} = (1)^2 p + (2)^2 pq + (3)^2 pq^2 + (4)^2 pq^3 + \cdots = E[X^2]$$

And $\frac{p(1-q^2)}{(1-q)^4} = \frac{p(1-q)(1+q)}{(1-q)^4} = \frac{p(p)(1+q)}{p^4} = \frac{1+q}{p^2}$.

Thus, $Var[X] = E[X^2] - (E[X])^2 = \frac{1+q}{p^2} - \left(\frac{1}{p}\right)^2 = \frac{q}{p^2}$