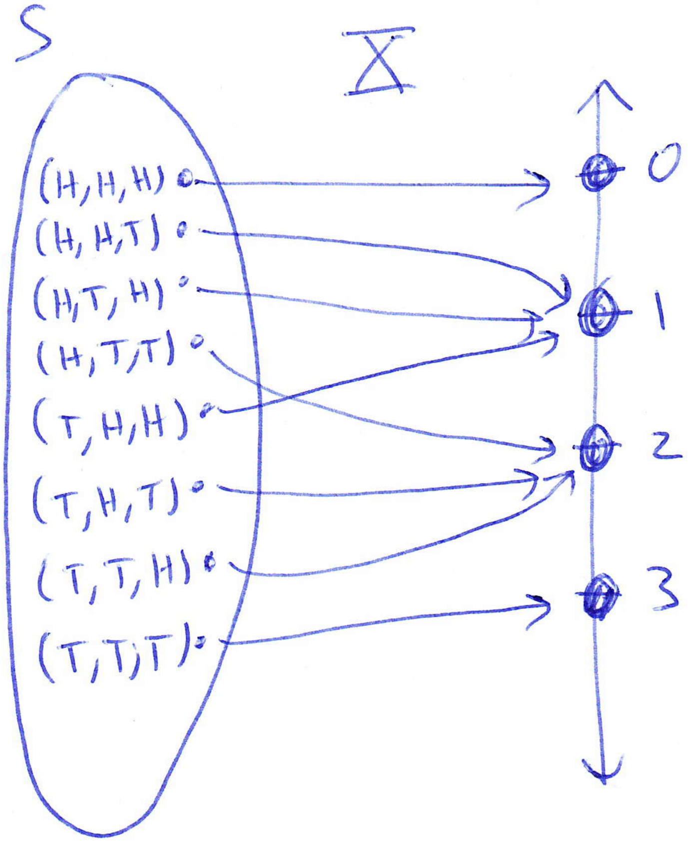
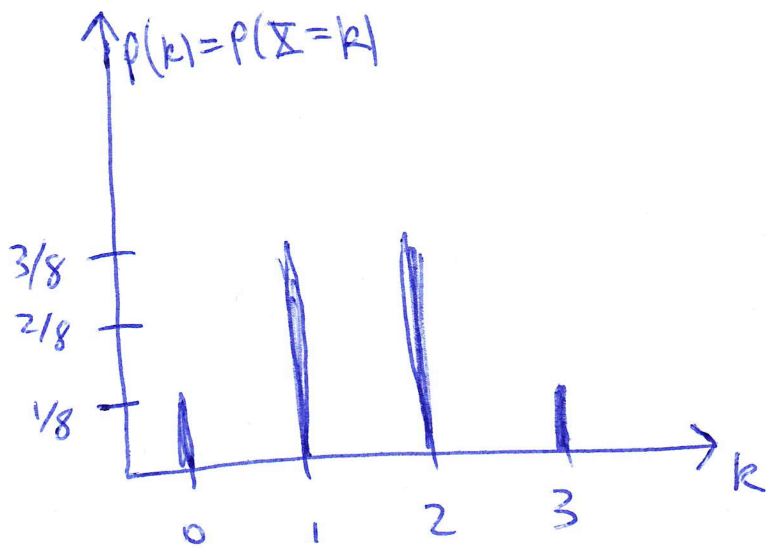
Homework #6 Solutions

(1)





$p(0) = P(X=0) = \frac{1}{8}$

$p(1) = P(X=1) = \frac{3}{8}$

$p(2) = P(X=2) = \frac{3}{8}$

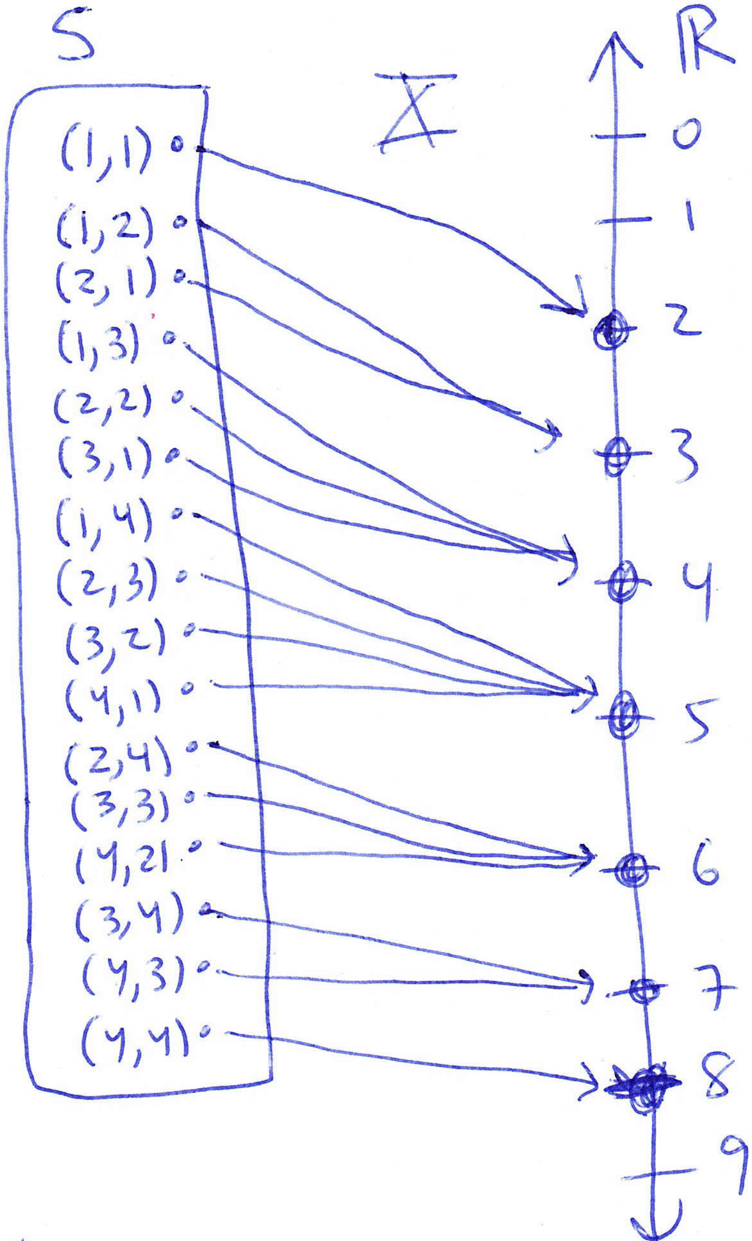
$p(3) = P(X=3) = \frac{1}{8}$

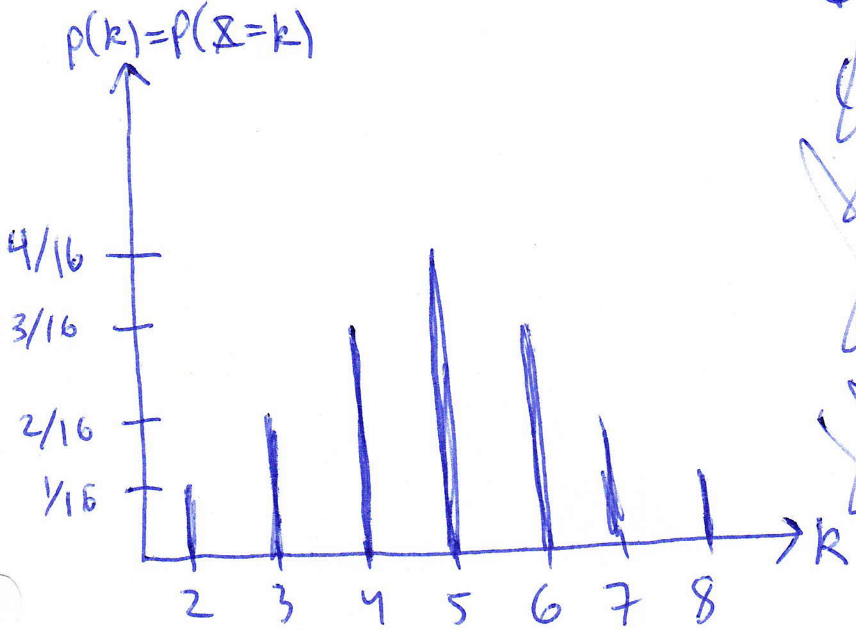
$$E[X] = 0\left(\frac{1}{8}\right) + (1)\left(\frac{3}{8}\right) + (2)\left(\frac{3}{8}\right) + (3)\left(\frac{1}{8}\right) = \frac{3+6+3}{8} = \frac{12}{8} = 1.5$$

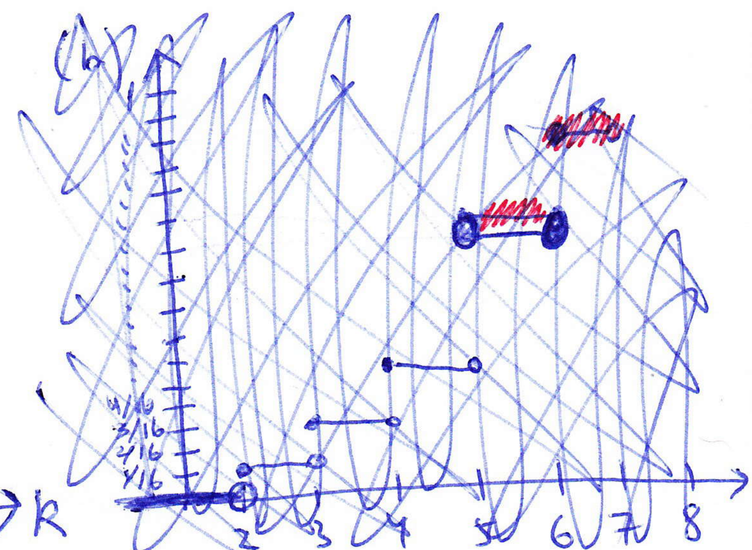
$$E[X^2] = 0^2\left(\frac{1}{8}\right) + (1)^2\left(\frac{3}{8}\right) + (2)^2\left(\frac{3}{8}\right) + (3)^2\left(\frac{1}{8}\right) = \frac{3+12+9}{8} = \frac{24}{8} = 3$$

$$\mathrm{Var}(X) = E[X^2] - (E[X])^2 = 3 - (1.5)^2 = 0.75$$

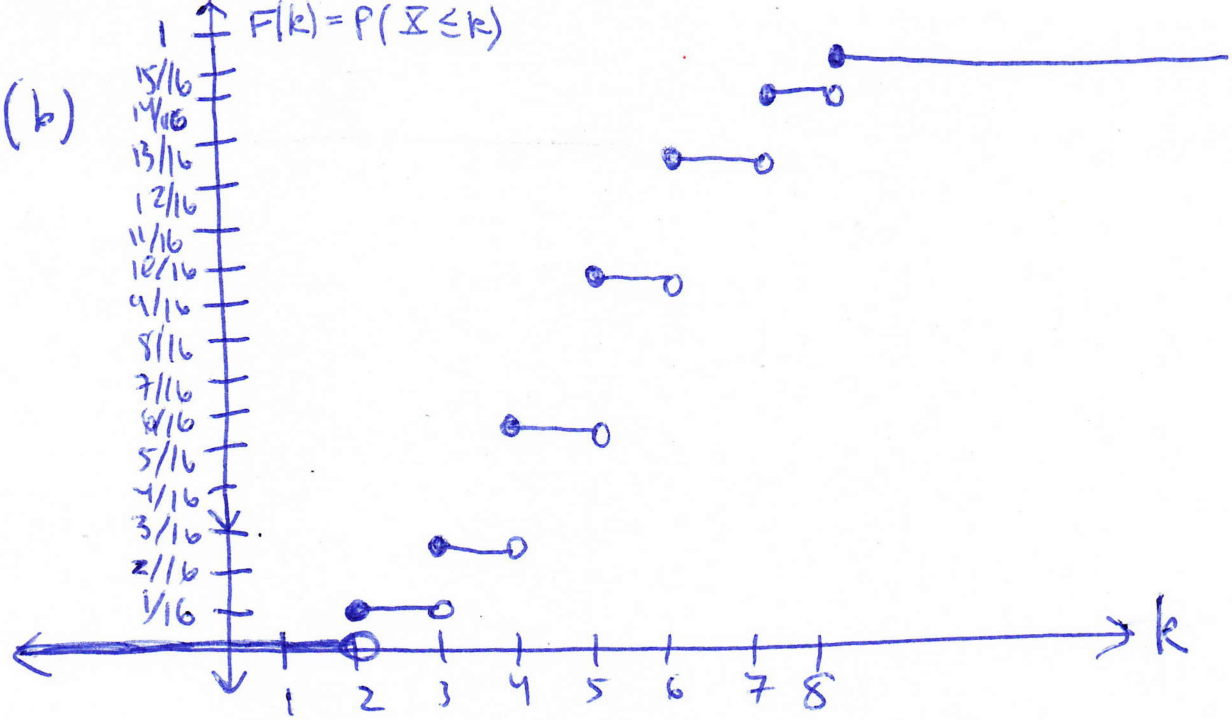
② (a)


S
X
R
(1,1)
(1,2)
(2,1)
(1,3)
(2,2)
(3,1)
(1,4)
(2,3)
(3,2)
(4,1)
(2,4)
(3,3)
(4,2)
(3,4)
(4,3)
(4,4)
0
1
2
3
4
5
6
7
8
9


p(k)=P(X=k)
4/16
3/16
2/16
1/16
k
2
3
4
5
6
7
8



(b)



(c) $E[X] = 2\left(\frac{1}{16}\right) + 3\left(\frac{2}{16}\right) + 4\left(\frac{3}{16}\right) + 5\left(\frac{4}{16}\right) + 6\left(\frac{3}{16}\right) + 7\left(\frac{2}{16}\right) + 8\left(\frac{1}{16}\right)$

$= \frac{2+6+12+20+18+14+8}{16} = \frac{80}{16} = 5$

$E[X^2] = 2^2\left(\frac{1}{16}\right) + 3^2\left(\frac{2}{16}\right) + 4^2\left(\frac{3}{16}\right) + 5^2\left(\frac{4}{16}\right) + 6^2\left(\frac{3}{16}\right) + 7^2\left(\frac{2}{16}\right) + 8^2\left(\frac{1}{16}\right)$

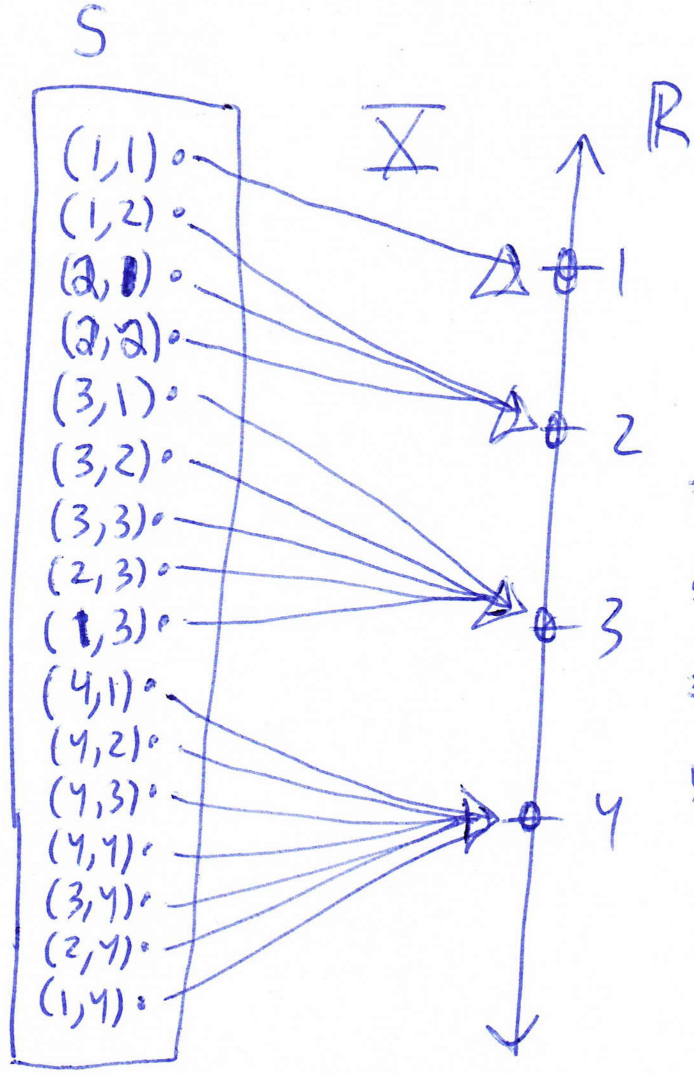
$= \frac{4+18+48+100+108+98+64}{16} = \frac{440}{16} = 27.5$

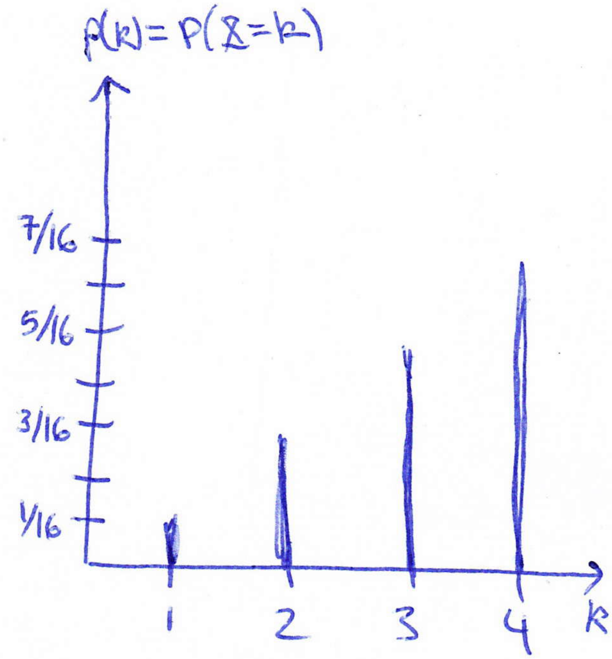
$\mathrm{Var}(X) = E[X^2] - (E[X])^2 = 27.5 - (5)^2 = 2.5$

$\sigma = \sqrt{2.5} \approx 1.58$

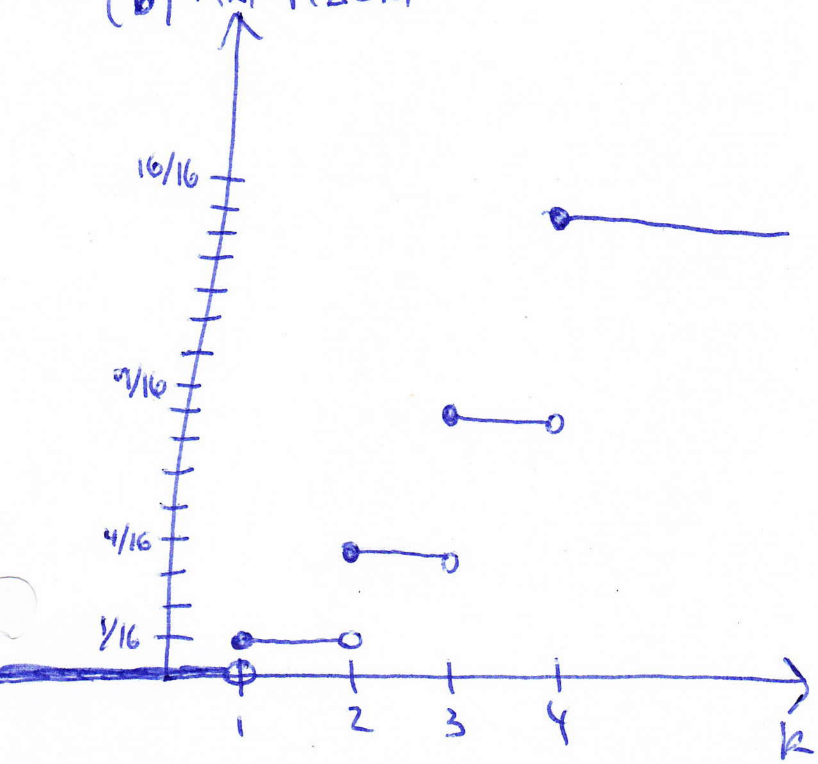
③

(a)





(b) $F(k) = P(X \le k)$



(c)

$$E[X] = (1)\left(\frac{1}{16}\right) + (2)\left(\frac{3}{16}\right) + (3)\left(\frac{5}{16}\right) + (4)\left(\frac{7}{16}\right)$$

$$= \frac{1+6+15+21}{16} = \frac{43}{16}$$

$$\approx 2.6875$$

$$E[X^2] = (1)^2\left(\frac{1}{16}\right) + (2)^2\left(\frac{3}{16}\right) + (3)^2\left(\frac{5}{16}\right) + (4)^2\left(\frac{7}{16}\right)$$

$$= \frac{1+12+45+112}{16}$$

$$= \frac{170}{16} \approx 10.625$$

$$Var[X] = E[X^2] - \left(E[X]\right)^2$$

$$= \frac{170}{16} - \left(\frac{43}{16}\right)^2$$

$$= \frac{2720 - 1849}{256} = \frac{871}{256}$$

$$\approx 3.40$$

$$\sigma = \sqrt{Var[X]} = \sqrt{\frac{871}{256}}$$

④ (a)

GAME A

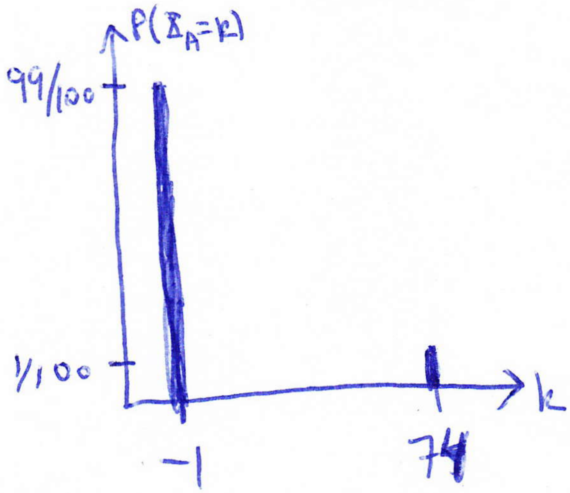
$P(X_A = -1) = \frac{99}{100}$

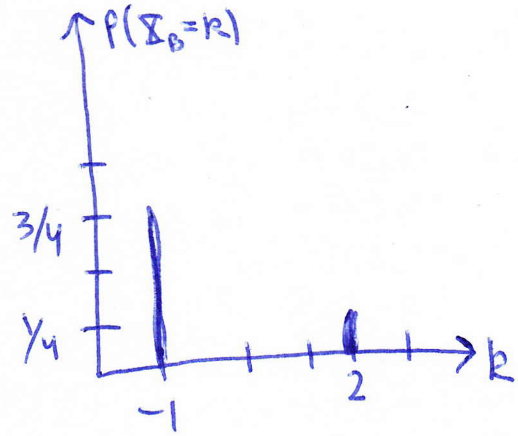
$P(X_A = 74) = \frac{1}{100}$

GAME B

$P(X_B = -1) = \frac{3}{4}$

$P(X_B = 2) = \frac{1}{4}$





(b) $E[X_A] = (-1)\left(\frac{99}{100}\right) + (74)\left(\frac{1}{100}\right) = \frac{-25}{100} = -\frac{1}{4}$

$E[X_A^2] = (-1)^2\left(\frac{99}{100}\right) + (74)^2\left(\frac{1}{100}\right) = \frac{99 + 5476}{100} = \frac{5575}{100} = 55.75$

$\mathrm{Var}(X_A) = E[X_A^2] - (E[X_A])^2 = \frac{5575}{100} - \left(-\frac{1}{4}\right)^2 = \frac{5575}{100} - \frac{1}{16} = \frac{89200 - 100}{1600}$

$= \frac{89100}{1600} \approx 55.6875$

Same expected value

(c) $E[X_B] = (-1)\left(\frac{3}{4}\right) + (2)\left(\frac{1}{4}\right) = \frac{-3+2}{4} = -\frac{1}{4}$

$E[X_B^2] = (-1)^2\left(\frac{3}{4}\right) + (2)^2\left(\frac{1}{4}\right) = \frac{3}{4} + \frac{4}{4} = \frac{7}{4}$

$\mathrm{Var}[X_B] = E[X_B^2] - (E[X_B])^2 = \frac{7}{4} - \left(-\frac{1}{4}\right)^2 = \frac{7}{4} - \frac{1}{16} = \frac{27}{16} \approx 1.038\ldots$

(d) Both games have the same expected value in the long run of -0.25 cents. Game A has bigger swings than game B. This is reflected in the variance. What kind of game would you want to play? It would be a personal preference. It also depends on how much money you have so you can ride out the swings of each game. However, in the long run, you will lose on both games.

5

(a) Let $t = k\sigma$ in Chebyshev's inequality.

(b) Let $k = 2$ in part (a).

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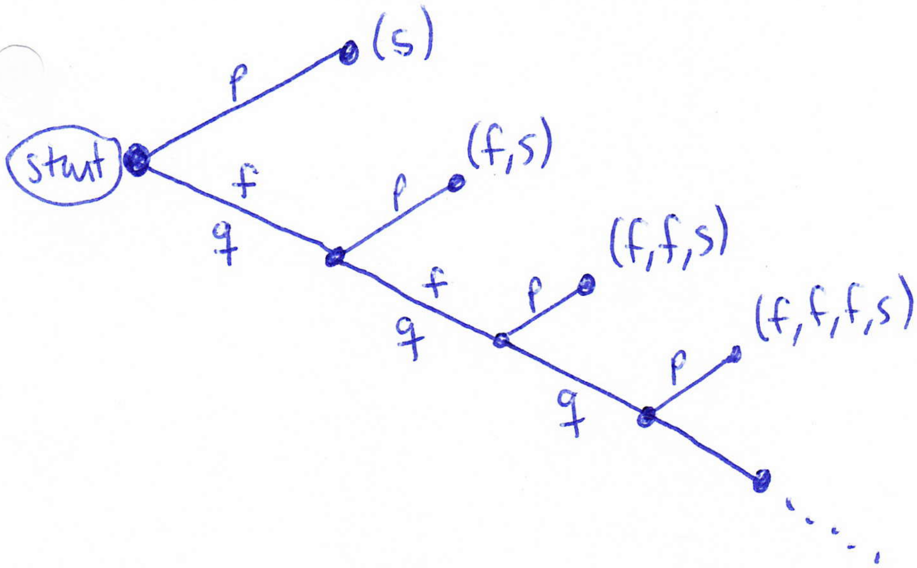
6 Let s represent success and
f represent failure,

(a) $S = \{ (s), (f,s), (f,f,s), (f,f,f,s), \ldots\ldots \}$

It's like this tree :



$P((s)) = p$

$P((f,s)) = qp$

$P((f,f,s)) = q^2 p$

$\vdots$

$P((\underbrace{f,f,\ldots,f}_{n \text{ f's}},s)) = q^n p$

Let X = # trials before first success occurs. Then,

$P(X=1) = p$

$P(X=2) = qp$

$P(X=3) = q^2 p$

$\vdots$

$P(X=k) = q^{k-1} p$ where $k \geq 1$.

(a) continued... Define P so that $P(E) = \sum_{\omega \in E} P(\{\omega\})$, and the values $P(\{\omega\})$ are as on the previous page.

Note that this is a probability space since

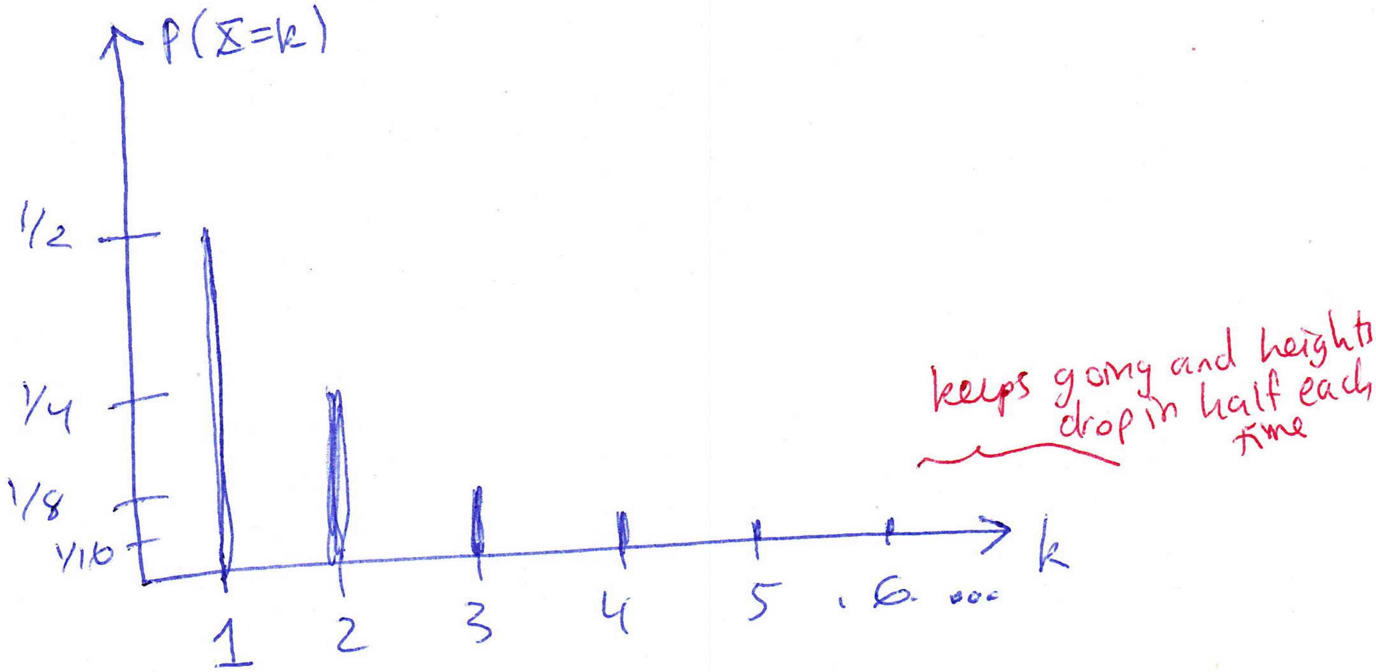
$$P(S) = \sum_{\omega \in S} P(\omega) = p + qp + q^2 p + q^3 p + \dots$$

$$= p\left[1 + q + q^2 + q^3 + \dots\right]$$

$$= p\left[\frac{1}{1-q}\right] = \frac{p}{1-q} = \frac{p}{p} = 1.$$

~~so everything is good.~~

(b) $p = \frac{1}{2}$, $q = \frac{1}{2}$, $P(X=k) = \left(\frac{1}{2}\right)^{k-1}\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^k$



(c) Recall that $P(X=k) = pq^{k-1}$

$$E[X] = (1)(p) + (2)(pq) + (3)(pq^2) + (4)(pq^3) + \cdots$$

Suppose that $|x|<1$. Then

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots$$

Differentiating both sides gives

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$$

Set $x = q$ which satisfies $|q|<1$.

This gives

$$\frac{1}{p^2} = \frac{1}{(1-q)^2} = 1 + 2q + 3q^2 + 4q^3 + \cdots$$

Now multiply by p to get

$$\frac{1}{p} = p + 2pq + 3pq^2 + 4pq^3 + \cdots = E[X].$$

Now let's compute $Var[X]$.

We need $E[X^2] = (1)^2 p + (2)^2 pq + (3)^2 pq^2 + (4)^2 pq^3 + \cdots$

From above, if $|x|<1$ then

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$$

So,

$$\frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + 4x^4 + \cdots$$

Differentiating gives

$$\frac{(1-x)^2 - x[2(1-x)(-1)]}{(1-x)^4} = 1 + (2^2)x + (3^2)x^2 + (4^2)x^3 + \cdots$$

So,

$$\frac{1-x^2}{(1-x)^4} = (1)^2 + (2)^2 x + (3)^2 x^2 + (4)^2 x^3 + \cdots$$

Set $q = x$ and multiply through by p to get

$$\frac{p(1-q^2)}{(1-q)^4} = (1)^2 p + (2)^2 pq + (3)^2 pq^2 + (4)^2 pq^3 + \cdots = E[X^2]$$

And $\frac{p(1-q^2)}{(1-q)^4} = \frac{p(1-q)(1+q)}{(1-q)^4} = \frac{p(p)(1+q)}{p^4} = \frac{1+q}{p^2}$.

Thus, $Var[X] = E[X^2] - (E[X])^2 = \frac{1+q}{p^2} - \left(\frac{1}{p}\right)^2 = \frac{q}{p^2}$