Homework #7 Solutions

① Let $X = \#$ of 4s occuring in $n=100$ rolls.
Let $p = \frac{1}{6}$. Then

(a)

$$P(0 \le X \le 15) \approx P\left(\frac{0-100(\frac{1}{6})}{\sqrt{100(\frac{1}{6})(\frac{5}{6})}} \le \frac{X-100(\frac{1}{6})}{\sqrt{100(\frac{1}{6})(\frac{5}{6})}} \le \frac{15-100(\frac{1}{6})}{\sqrt{100(\frac{1}{6})(\frac{5}{6})}}\right)$$

$$\approx \Phi(-0.44721\ldots) - \Phi(-4.472136\ldots)$$

$$\approx \left[1-\Phi(0.44721\ldots)\right] - \left[1-\underbrace{\Phi(4.472136\ldots)}_{\approx 1}\right]$$

$$\approx 1 - 0.67 \approx 0.33$$

(b) $P(X=15) = P(14.5 \le X \le 15.5)$

$$= P\left(\frac{14.5-100(\frac{1}{6})}{\sqrt{100(\frac{1}{6})(\frac{5}{6})}} \le \frac{X-100(\frac{1}{6})}{\sqrt{100(\frac{1}{6})(\frac{5}{6})}} \le \frac{15.5-100(\frac{1}{6})}{\sqrt{100(\frac{1}{6})(\frac{5}{6})}}\right)$$

$$\approx \Phi(-0.3130495\ldots) - \Phi(-0.5813767\ldots)$$

$$\approx (1-\Phi(0.313)) - (1-\Phi(0.581))$$

$$\approx \Phi(0.581) - \Phi(0.313) \approx 0.7190 - 0.6217 \approx 0.0973$$

② Let X be a Poisson random variable with parameter $\lambda > 0$.

(a) $$E[X] = \sum_{k=0}^{\infty} k P(X=k) = \sum_{k=0}^{\infty} k \frac{e^{-\lambda}\lambda^k}{k!}$$

$$= e^{-\lambda} \sum_{k=1}^{\infty} \frac{k\lambda^k}{k!} = \lambda e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} = \lambda e^{-\lambda} \sum_{l=0}^{\infty} \frac{\lambda^l}{l!}$$

$$= \lambda e^{-\lambda} e^{\lambda} = \lambda.$$

$$\boxed{e^x = \sum_{l=0}^{\infty} \frac{x^l}{l!} \qquad e^{\lambda} = \sum_{l=0}^{\infty} \frac{\lambda^l}{l!}}$$

(b) $$E[X^2] = \sum_{k=0}^{\infty} k^2 P(X=k)$$

$$= \sum_{k=0}^{\infty} k^2 \frac{e^{-\lambda}\lambda^k}{k!} = \lambda e^{-\lambda} \sum_{k=1} \frac{k\lambda^{k-1}}{(k-1)!}$$

$$= \lambda e^{-\lambda} \sum_{l=0}^{\infty} \frac{(l+1)\lambda^l}{l!} = \lambda e^{-\lambda} \Big[\underbrace{\sum_{l=0}^{\infty} \frac{l\lambda^l}{l!}}_{\lambda e^{\lambda} \text{ (as calculated above)}} + \underbrace{\sum_{l=0}^{\infty} \frac{\lambda^l}{l!}}_{e^{\lambda}} \Big]$$

$\boxed{l = k-1}$

$$= \lambda e^{-\lambda} \left[\lambda e^{\lambda} + e^{\lambda} \right] = \lambda[\lambda + 1].$$

So, $\mathrm{Var}(X) = E[X^2] - (E[X])^2 = \lambda(\lambda+1) - \lambda^2 = \lambda$

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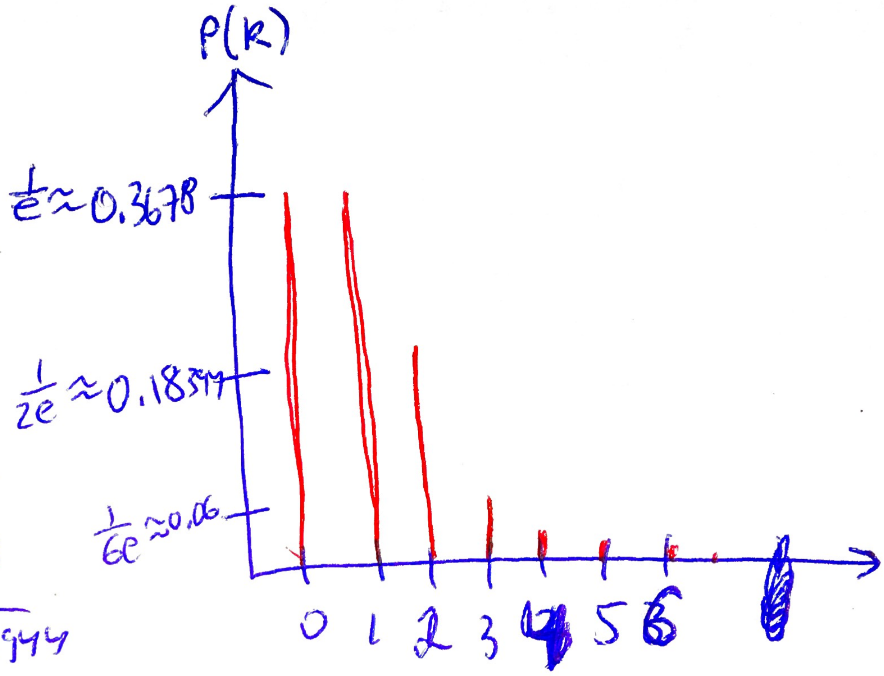
Let $\lambda = 1$. Then $p(k) = P(X=k) = \frac{e^{-1}\cdot(1)^k}{k!} = \frac{1}{e\cdot k!}$

k	$p(k) = \frac{1}{e\cdot k!}$
0	$\frac{1}{e} \approx 0.367879$
1	$\frac{1}{e} \approx 0.367879$
2	$\frac{1}{2e} \approx 0.18394$
3	$\frac{1}{6e} \approx 0.0613132$
4	$\frac{1}{24e} \approx 0.0153283$
5	$\frac{1}{120e} \approx 0.00306566$
6	$\frac{1}{720e} \approx 0.000510944$



If $X<1$, there is only the case that $X=0$

(4) Let X be the binomial distribution with $n=20$ and $p=0.01$ (here $X = \#$ incorrect bills)

Then $P(X \geq 1) = 1 - P(X<1) = 1 - P(X=0)$

$= 1 - \binom{20}{0}(0.01)^0(0.99)^{20} \approx 1 - 0.8179$

$\approx 0.182093\ldots$

Poisson approximation

$$1 - P(X=0) \approx 1 - \frac{[(20)(0.01)]^0}{0!} e^{-20(0.01)}$$

$\boxed{\lambda = np}$

$\approx 1 - 0.8187\ldots \approx 0.1813$

(5) Here we have $n = 50$ independent trials each with success rate $p = \frac{1}{100}$

So this is a binomial random variable

We approximate this with the Poisson random variable.

Let X be the binomial random variable with $n = 50$ and $p = \frac{1}{100}$.

Let $\lambda = np = \frac{50}{100} = \frac{1}{2}$.

$k = 0$
$\lambda = 1/2$

(a) $P(X \geq 1) = 1 - P(X = 0)$

$$\approx 1 - \frac{(\frac{1}{2})^0}{0!} e^{-1/2}$$

$$= 1 - e^{-1/2} \approx 0.393469$$

$k = 1$
$\lambda = 1/2$

(b) $P(X = 1) \approx \frac{(\frac{1}{2})^1}{1!} e^{-(\frac{1}{2})} = \frac{1}{2} e^{-1/2} \approx 0.303265$

(c) $P(X \geq 2) = 1 - P(X < 2)$

$$= 1 - P(X = 0) - P(X = 1)$$

$$= 1 - e^{-1/2} - \frac{1}{2} e^{-1/2}$$

$$\approx 0.090204\ldots$$