Math 5401 - Test 1

Name: Solutions

Score	
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2	
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1. [10 points - 3,3,4]

(a) Simplify $sr^2sr^3(r^2s)^{-1}$ in $D_8 = \{1, r, r^2, r^3, s, sr, sr^2, sr^3\}$.

$$sr^2sr^3(r^2s)^{-1} = sr^2sr^3s^{-1}r^{-2} = sr^2sr^3sr^2 = s^2r^{-2}r^3sr^2 = 1 \cdot r \cdot s \cdot r^2$$
$$= rsr^2 = s\,r^{-1}r^2 = \boxed{sr}$$

(b) Compute the cyclic subgroup generated by $\overline{5}$ in $\mathbb{Z}_{20}$. What is the order of $\overline{5}$?

$\langle \overline{5} \rangle = \{\overline{0}, \overline{5}, \overline{10}, \overline{15}\}$

order of $\overline{5}$ is 4.

(c) Consider the subset $H = \{1, r^2, s, sr^2\}$ of D_8. Is H a subgroup of D_8? Verify your answer.

H	1	r^2	s	sr^2
1	1	r^2	s	sr^2
r^2	r^2	1	sr^2	s
s	s	sr^2	1	r^2
sr^2	sr^2	s	r^2	1

By the table H is closed. And H is closed under inversion.

And $1 \in H$.

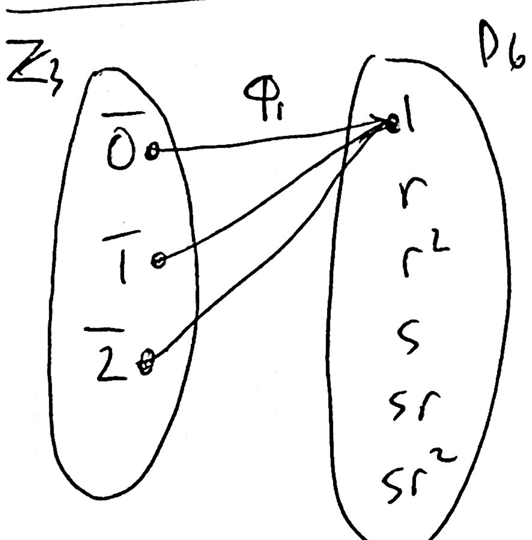
So, $H \leq D_8$.

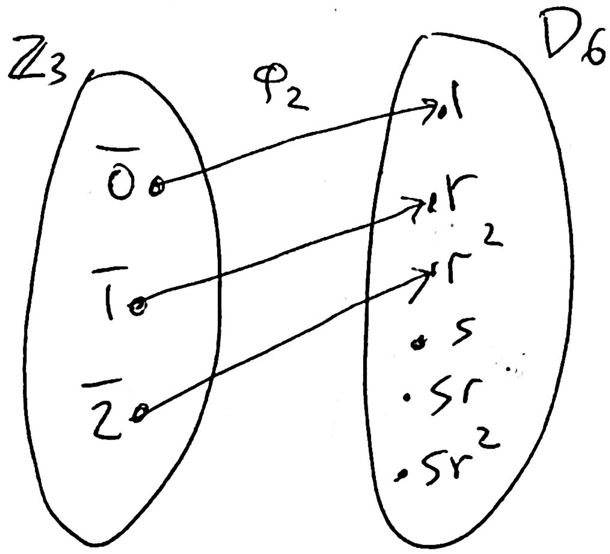
2. [6 points] Find all homomorphisms $\phi : \mathbb{Z}_3 \to D_6$. Draw a picture of each one.

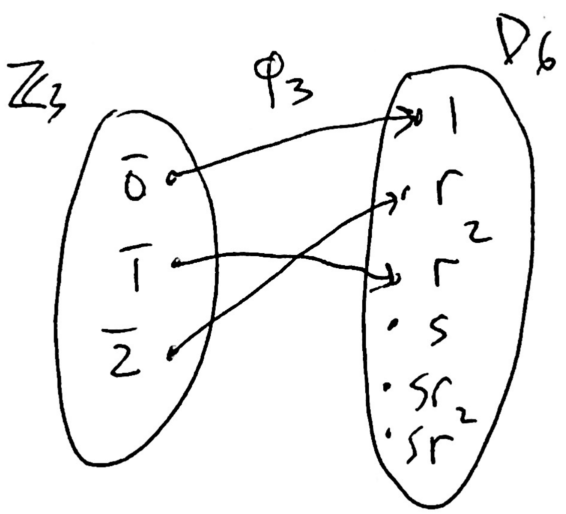
$\bar{1}$ has order 3 in $\mathbb{Z}_3$

element	order in D6
1	1
r	3
r^2	3
s	2
sr	2
sr^2	2

orders dividing 3







3. [15 points - 5 each] Let G and H be groups. Let $\phi : G \to H$ be a group homomorphism. Let 1_G and 1_H be the identity elements of G and H.

(a) Prove that $\phi(1_G) =$ [redacted] 1_H

(b) Prove that $\phi(x^{-1}) = \phi(x)^{-1}$ for all $x \in G$.

(c) Prove that $\ker(\phi)$ is a subgroup of G.

(a) Note that $\varphi(1_G) = \varphi(1_G \cdot 1_G) = \varphi(1_G)\varphi(1_G)$.
Applying $\varphi(1_G)^{-1}$ to both sides gives

$$\varphi(1_G)^{-1}\varphi(1_G) = \varphi(1_G)^{-1}\varphi(1_G)\varphi(1_G).$$

So, $1_H = 1_H\,\varphi(1_G)$.

Thus, $1_H = \varphi(1_G)$.

(b) Note that

$$\varphi(x)\varphi(x^{-1}) = \varphi(xx^{-1}) = \varphi(1_G) = 1_H$$

and

$$\varphi(x^{-1})\varphi(x) = \varphi(x^{-1}x) = \varphi(1_G) = 1_H.$$

So,

$$\varphi(x^{-1}) = [\varphi(x)]^{-1}.$$

(c)

- $1_G \in \ker(\varphi)$ since by (a) we have that $\varphi(1_G) = 1_H$.
- Suppose $x, y \in \ker(\varphi)$.

  Then $\varphi(x) = 1_H$ and $\varphi(y) = 1_H$.

  So, $\varphi(y^{-1}) = \varphi(y)^{-1} = 1_H^{-1} = 1_H$.

  Thus,

  $$\varphi(xy^{-1}) = \varphi(x)\varphi(y^{-1}) = 1_H \cdot 1_H = 1_H.$$

Therefore, $xy^{-1} \in \ker(\varphi)$.

So, $\ker(\varphi) \leq G$.

4. [8 points] PICK ONE OF THE FOLLOWING: (You may only choose one. Circle the one you picked. If you do both then I will grade A.)

A) Let G and H be groups. Let 1_G and 1_H be the identity elements of G and H respectively. Prove that $G \times H$ is a group under the usual operation $(a,b)(c,d) = (ac, bd)$. [That is, check the four group properties.]

B) Let G be a group and $\phi : G \to G$ be defined by $\phi(g) = g^2$. Prove that G is abelian if and only if ϕ is a homomorphism.

A)

- Let $a,c \in G$ and $b,d \in H$. Since G is a group $ac \in G$.
  Since H is a group $bd \in H$.
  So, $(a,b)(c,d) = (ac, bd) \in G \times H$.
- Let $(a,b),(c,d),(e,f) \in G \times H$.
  Since G is a group $a(ce) = (ac)e$.
  Since H is a group $b(df) = (bd)f$.
  So, $(a,b)[(c,d)(e,f)] = (a,b)(ce,df)$
  $= (a(ce), b(df)) = ((ac)e, (bd)f)$
  $= [(a,b)(c,d)](e,f)$.
- $(1_G, 1_H)$ is the identity of $G \times H$, since
  $(1_G, 1_H)(a,b) = (1_G a, 1_H b) = (a,b)$
  and $(a,b)(1_G, 1_H) = (a 1_G, b 1_H) = (a,b)$
  for all $(a,b) \in G \times H$.
- Let $(a,b) \in G \times H$. Since $a \in G$ and G is a group $a^{-1} \in G$ exists.
  Since $b \in H$ and H is a group, $b^{-1} \in H$ exists. And $(a,b)^{-1} = (a^{-1}, b^{-1})$
  since $(a,b)(a^{-1}, b^{-1}) = (aa^{-1}, bb^{-1}) = (1_G, 1_H)$
  and $(a^{-1}, b^{-1})(a,b) = (a^{-1}a, b^{-1}b) = (1_G, 1_H)$.

4. [8 points] PICK ONE OF THE FOLLOWING: (You may only choose one. Circle the one you picked. If you do both then I will grade A.)

A) Let G and H be groups. Let 1_G and 1_H be the identity elements of G and H respectively. Prove that $G \times H$ is a group under the usual operation $(a,b)(c,d) = (ac, bd)$. [That is, check the four group properties.]

B) Let G be a group and $\phi : G \to G$ be defined by $\phi(g) = g^2$. Prove that G is abelian if and only if ϕ is a homomorphism.

B)

($\Rightarrow$) Suppose G is abelian.
Let $a, b \in G$.
Then

$$\varphi(ab) = (ab)^2 = (ab)(ab) = abab \underset{\substack{\uparrow \\ \boxed{G \text{ is abelian}}}}{=} aabb = a^2b^2 = \varphi(a)\varphi(b).$$

So, φ is a homomorphism.

($\Leftarrow$) Suppose φ is a homomorphism
Let $a, b \in G$.
Then, $\varphi(ab) = \varphi(a)\varphi(b)$.
So, $(ab)^2 = a^2b^2$,
That is, $abab = aabb$.
So, $a^{-1}(abab)b^{-1} = a^{-1}(aabb)b^{-1}$
Thus, $ba = ab$.
So, G is abelian.

5. [8 points] PICK ONE OF THE FOLLOWING: (You may only choose one. Circle the one you picked. If you do both then I will grade A.)

A) Let G be a group and $H \leq G$. Suppose that $|H| = 2$, that is $H = \{1, x\}$ for some $x \in G$.

(a) Prove that $N_G(H) = C_G(H)$.

(b) Prove that if $N_G(H) = G$ then $H \leq Z(G)$.

B) Let G_1, G_2, H_1, and H_2 be groups. Prove that if $G_1 \cong G_2$ and $H_1 \cong H_2$, then $G_1 \times H_1 \cong G_2 \times H_2$.

A) Recall that

$N_G(A) = \{g \in G \mid gAg^{-1} = A\}$

$C_G(A) = \{g \in G \mid ga = ag \text{ for all } a \in A\}$.

(a) Let's show that $N_G(H) \subseteq C_G(H)$ and $C_G(H) \subseteq N_G(H)$.

$\boxed{\subseteq}$ Let $g \in N_G(H)$. then $gHg^{-1} = H$. So, $\{g1g^{-1}, gxg^{-1}\} = \{1, x\}$. Thus, $\{1, gxg^{-1}\} = \{1, x\}$. So, $gxg^{-1} = x$. So, $gx = xg$. Thus, $g \in C_G(H)$. Thus, $N_G(H) \subseteq C_G(H)$.

$\boxed{\supseteq}$ Now let $g \in C_G(H)$. Then $gh = hg$ for all $h \in H$. Thus, $gHg^{-1} = \{ghg^{-1} \mid h \in H\} = \{hgg^{-1} \mid h \in H\} = \{h \mid h \in H\} = H$. So, $g \in N_G(H)$.

Therefore $C_G(H) = N_G(H)$.

(next page for (b))

5. [8 points] PICK ONE OF THE FOLLOWING: (You may only choose one. Circle the one you picked. If you do both then I will grade A.)

A) Let G be a group and $H \leq G$. Suppose that $|H| = 2$, that is $H = \{1, x\}$ for some $x \in G$.

(a) Prove that $N_G(H) = C_G(H)$.

(b) Prove that if $N_G(H) = G$ then $H \leq Z(G)$.

B) Let G_1, G_2, H_1, and H_2 be groups. Prove that if $G_1 \cong G_2$ and $H_1 \cong H_2$, then $G_1 \times H_1 \cong G_2 \times H_2$.

A) continued...

(b) Suppose that $N_G(H) = G$.

Then, $gHg^{-1} = H$ for all $g \in G$.

So, $g\{1, x\}g^{-1} = \{1, x\}$ for all $g \in G$.

Thus, $\{g1g^{-1}, gxg^{-1}\} = \{1, x\}$ for all $g \in G$.

So, $\{1, gxg^{-1}\} = \{1, x\}$ for all $g \in G$.

So, $gxg^{-1} = x$ for all $g \in G$.

So, $gx = xg$ for all $g \in G$.

So, $x \in Z(G)$. Also $1 \in Z(G)$.

Thus, $H = \{1, x\} \subseteq Z(G)$. $\blacksquare$

5. [8 points] PICK ONE OF THE FOLLOWING: (You may only choose one. Circle the one you picked. If you do both then I will grade A.)

A) Let G be a group and $H \leq G$. Suppose that $|H| = 2$, that is $H = \{1, x\}$ for some $x \in G$.

(a) Prove that $N_G(H) = C_G(H)$.

(b) Prove that if $N_G(H) = G$ then $H \leq Z(G)$.

B) Let G_1, G_2, H_1, and H_2 be groups. Prove that if $G_1 \cong G_2$ and $H_1 \cong H_2$, then $G_1 \times H_1 \cong G_2 \times H_2$.

B) Since $G_1 \cong G_2$ there exists an isomorphism $\psi_1 : G_1 \to G_2$. Since $H_1 \cong H_2$ there exists an isomorphism $\psi_2 : H_1 \to H_2$.

Define $\psi : G_1 \times H_1 \longrightarrow G_2 \times H_2$ by $\psi(a,b) = (\psi_1(a), \psi_2(b))$.

- ψ is an ~~Hom~~ homomorphism

Let $(a,b),(c,d) \in G_1 \times H_1$. Then,

$$\psi((a,b)(c,d)) = \psi(ac, bd) = (\psi_1(ac), \psi_2(bd))$$
$$= (\psi_1(a)\psi_1(c), \psi_2(b)\psi_2(d))$$

since ψ_1 and ψ_2 are isomorphisms

$$= (\psi_1(a), \psi_2(b))(\psi_1(c), \psi_2(d))$$
$$= \psi(a,b)\psi(c,d).$$

- ψ is 1-1

Suppose that $\psi(a,b) = \psi(c,d)$ for some $(a,b),(c,d) \in G_1 \times H_1$. Then $(\psi_1(a), \psi_2(b)) = (\psi_1(c), \psi_2(d))$. So, $\psi_1(a) = \psi_1(c)$ and $\psi_2(b) = \psi_2(d)$.

(next page)

5. [8 points] PICK ONE OF THE FOLLOWING: (You may only choose one. Circle the one you picked. If you do both then I will grade A.)

A) Let G be a group and $H \leq G$. Suppose that $|H| = 2$, that is $H = \{1, x\}$ for some $x \in G$.

(a) Prove that $N_G(H) = C_G(H)$.

(b) Prove that if $N_G(H) = G$ then $H \leq Z(G)$.

B) Let G_1, G_2, H_1, and H_2 be groups. Prove that if $G_1 \cong G_2$ and $H_1 \cong H_2$, then $G_1 \times H_1 \cong G_2 \times H_2$.

B) continued...

Since ψ_1 is 1-1, $a = c$.
Since ψ_2 is 1-1, $b = d$.
So, $(a,b) = (c,d)$,

ψ is onto.

Let $(x,y) \in G_2 \times H_2$.
Since ψ_1 is onto there exists $a \in G_1$ with $\psi_1(a) = x$.
Since ψ_2 is onto there exists $b \in H_1$ with $\psi_2(b) = y$,
So, $(a,b) \in G_1 \times H_1$ and $\psi(a,b) = (\psi_1(a), \psi_2(b)) = (x,y)$.