Math 5402 - Test 1

Note: Let R and S be rings. Recall that addition and multiplication in $R \times S$ is given by $(a,b)+(c,d)=(a+c,b+d)$ and $(a,b)(c,d)=(ac,bd)$.

Name: Solutions

Score	
1 (a,b)	
1 (c)	
1 (d)	
2	
3	
4	
T	

1. [32 points - 8 each]

(a) ~~Prove that~~ IS $\phi : \mathbb{Z} \to \mathbb{Z}\times\mathbb{Z}$ given by $\phi(x) = (2x, x)$ a ring homomorphism. ?

No.

$$\varphi((1,1)(1,1)) = \varphi(1,1) = (2,1)$$

but

$$\varphi(1,1)\varphi(1,1) = (2,1)(2,1) = (4,1)$$

(b) Is $\{\bar{0}, \bar{4}, \bar{8}\}$ a prime ideal of $\mathbb{Z}_{12}$? Why or why not? (You may assume that it is an ideal.)

No.

$$\bar{2}\cdot\bar{2} = \bar{4} \in \mathbb{Z}_{12}$$

but $\bar{2} \notin \mathbb{Z}_{12}$

(c) Let $I = \{(-a, 3b) \mid a, b \in \mathbb{Z}\}$. Prove that I an ideal of $\mathbb{Z} \times \mathbb{Z}$.

1 $(0,0) = (-0, 3\cdot 0) \in I$

2 Let $x, y \in I$.

Then $x = (-a_1, 3b_1)$, $y = (-a_2, 3b_2)$

where $a_1, b_1, a_2, b_2 \in \mathbb{Z}$.

Then $x - y = (-(a_1 - a_2), 3(b_1 - b_2)) \in I$

3 Let $r \in \mathbb{Z} \times \mathbb{Z}$ and $x \in I$.

Then $r = (m, n)$ and

$x = (-a, 3b)$ where $m, n, a, b \in \mathbb{Z}$.

So, $rx = (-ma, 3nb) = (-(ma), 3(nb)) \in I$

and $xr = rx \in I$.

By 1, 2, 3 I is an ideal of

$\mathbb{Z} \times \mathbb{Z}$.

(d) Let F be a field. Can F have any zero divisors? Prove or disprove your answer.

No.

Suppose $ab = 0$ where $a, b \in F$.

case 1: If $a \neq 0$, then a^{-1} exists in F

and thus, $a^{-1}ab = a^{-1}0$.

Then $b = 0$,

case 2: If $b \neq 0$, then b^{-1} exists in F

and thus, $abb^{-1} = 0b^{-1}$.

So, $a = 0$,

Thus, we must have either $a = 0$

or $b = 0$.

So, F has no zero divisors.

2. [10 points] PICK ONE OF THE FOLLOWING:

A) Let R be a commutative ring with $1 \neq 0$, and let P an ideal of R with $P \neq R$. Prove that P is a prime ideal of R if and only if R/P is an integral domain.

B) Let R be a ring with identity $1 \neq 0$. Prove the following: (a) Let I be an ideal of R. Then $I = R$ if and only if I contains a unit of R. (b) Further suppose that R is commutative. Then, R is a field if the only ideals of R are $\{0\}$ and R.

These were done in class.
See study guide & notes.

A) See 2/12/20 notes

B) See 2/3/20 notes

3. [10 each] PICK ONE OF THE FOLLOWING:

A) Let R be a ring and I and J be ideals of R. Let

$$I + J = \{a + b \mid a \in I \text{ and } b \in J\}.$$

Prove that $I + J$ is an ideal of R.

B) Let $\phi : R \to S$ be a ring homomorphism where R and S are integral domains. Prove: (i) If I is an ideal of S, then $\phi^{-1}(I)$ is an ideal of R, and (ii) If P is a prime ideal of S, then $\phi^{-1}(P)$ is a prime ideal of R. [Assume $\varphi^{-1}(P) \neq R$]

(A)

(1) Since I and J are ideals, $0 \in I$ and $0 \in J$.
Hence $0 = 0 + 0 \in I + J$.

(2) Let $x, y \in I + J$. Then $x = a_1 + b_1$ and $y = a_2 + b_2$ where $a_1, a_2 \in I$ and $b_1, b_2 \in J$.
Then, $a_1 - a_2 \in I$ and $b_1 - b_2 \in J$ since I and J are ideals.
Thus,
$$x - y = (a_1 - a_2) + (b_1 - b_2) \in I + J.$$

(3) Let $r \in R$ and $z \in I + J$.
Then $z = \bar{i} + \bar{j}$ where $\bar{i} \in I$ and $\bar{j} \in J$.
Since I is an ideal, $r\bar{i} \in I$ and $\bar{i}r \in I$.
Since J is an ideal, $r\bar{j} \in J$ and $\bar{j}r \in J$.
So, $rz = r\bar{i} + r\bar{j} \in I + J$ and $zr = \bar{i}r + \bar{j}r \in I + J$.

By (1), (2), and (3) I is an ideal of R.

3. [10 each] PICK ONE OF THE FOLLOWING:

A) Let R be a ring and I and J be ideals of R. Let

$$I + J = \{a + b \mid a \in I \text{ and } b \in J\}.$$

Prove that $I + J$ is an ideal of R.

B) Let $\phi : R \to S$ be a ring homomorphism where R and S are integral domains. Prove: (i) If I is an ideal of S, then $\phi^{-1}(I)$ is an ideal of R, and (ii) If P is a prime ideal of S, then $\phi^{-1}(P)$ is a prime ideal of R. [Assume $\varphi^{-1}(P) \neq R$].

B

(i) $\varphi^{-1}(I) = \{r \in R \mid \varphi(r) \in I\}$.

- Since $\varphi(0) = 0 \in I$ we have that $0 \in \varphi^{-1}(I)$.
- Let $a, b \in \varphi^{-1}(I)$. Then $\varphi(a), \varphi(b) \in I$. Since I is an ideal, $\varphi(a) - \varphi(b) \in I$. So, $\varphi(a-b) \in I$. Thus, $a - b \in \varphi^{-1}(I)$.
- Let $c \in \varphi^{-1}(I)$ and $r \in R$. Since $c \in \varphi^{-1}(I)$ we get that $\varphi(c) \in I$. Since I is an ideal, $\varphi(r)\varphi(c) \in I$ and $\varphi(c)\varphi(r) \in I$. Thus, $\varphi(rc) \in I$ and $\varphi(cr) \in I$. So, $rc \in \varphi^{-1}(I)$ and $cr \in \varphi^{-1}(I)$.

Thus, $\varphi^{-1}(I)$ is an ideal of R.

(ii) Suppose $ab \in \varphi^{-1}(P)$. Then $\varphi(ab) \in P$. So, $\varphi(a)\varphi(b) \in P$. Since P is prime, either $\varphi(a) \in P$ or $\varphi(b) \in P$. Hence either $a \in \varphi^{-1}(P)$ or $b \in \varphi^{-1}(P)$. Thus, $\varphi^{-1}(P)$ is a prime ideal, since $\varphi^{-1}(P) \neq R$.

4. [10 points] PICK <u>ONE</u> OF THE FOLLOWING:

A) Let $\phi : F \to R$ be a ring homomorphism where F is a field and R is a ring. Prove: (i) The kernel $\ker(\phi)$ is an ideal of F, and (ii) If ϕ is onto and $R \neq \{0\}$ then ϕ is an isomorphism.

B) Let R and S be commutative rings with identities 1_R and 1_S respectively. (a) If A is an ideal of R and B is an ideal of S, show that $A \times B$ is an ideal of $R \times S$. (b) Show that every ideal I or $R \times S$ has the form $I = A \times B$ where A is an ideal of R and B is an ideal of S. [Hint for b: Define $A = \{a \in R \mid (a, 0) \in I\}$ and $B = \{b \in S \mid (0, b) \in I\}$.

A

(i)

- $\varphi(0) = 0$, so $0 \in \ker(\varphi)$
- Let $x, y \in \ker(\varphi)$. Then $\varphi(x) = \varphi(y) = 0$.
  So, $\varphi(x-y) = \varphi(x) - \varphi(y) = 0 - 0 = 0$
  So, $x - y \in \ker(\varphi)$.
- Let $r \in F$ and $z \in \ker(\varphi)$. Then $\varphi(z) = 0$.
  So, $\varphi(rz) = \varphi(r)\varphi(z) = \varphi(r) \cdot 0 = 0$
  and $\varphi(zr) = \varphi(z)\varphi(r) = 0\,\varphi(r) = 0$
  So, $rz \in \ker(\varphi)$ and $zr \in \ker(\varphi)$.

Thus, $\ker(\varphi)$ is an ideal of F.

(ii) Since $\ker(\varphi)$ is an ideal of F and F is a field, either $\ker(\varphi) = \{0\}$ or $\ker(\varphi) = F$. We know $\ker(\varphi) \neq F$. Why? If so, then $\varphi(x) = 0$ $\forall x \in F$. But $\operatorname{im}(\varphi) = \{0\}$ since φ is onto. $\operatorname{im}(\varphi) = R \neq \{0\}$ since
Therefore, $\ker(\varphi) = \{0\}$ and φ is 1-1. Since φ is also onto by assumption, φ is an isomorphism.

4. [10 points] PICK ONE OF THE FOLLOWING:

A) Let $\phi : F \to R$ be a ring homomorphism where F is a field and R is a ring. Prove: (i) The kernel $\ker(\phi)$ is an ideal of F, and (ii) If ϕ is onto and $R \neq \{0\}$ then ϕ is an isomorphism.

B) Let R and S be commutative rings with identities 1_R and 1_S respectively. (a) If A is an ideal of R and B is an ideal of S, show that $A \times B$ is an ideal of $R \times S$. (b) Show that every ideal I or $R \times S$ has the form $I = A \times B$ where A is an ideal of R and B is an ideal of S. [Hint for b: Define $A = \{a \in R \mid (a, 0) \in I\}$ and $B = \{b \in S \mid (0, b) \in I\}$.

B)

(a) • Since A and B are ideals, $0_R \in A$ and $0_S \in B$.
Hence $(0_R, 0_S) \in A \times B$.

• Let $x, y \in A \times B$. Then $x = (a_1, b_1)$ and $y = (a_2, b_2)$ are in $A \times B$. Since A is an ideal $\boxed{\text{of } R}$, $a_1 - a_2 \in A$.
Since B is an ideal $\boxed{\text{of } S}$, $b_1 - b_2 \in B$.
So, $x - y = (a_1 - a_2, b_1 - b_2) \in A \times B$.

• Let $m \in R \times S$ and $z \in A \times B$. Then $m = (r, s)$ and $z = (a, b)$ where $r \in R, s \in S, a \in A, b \in B$.
Since A is an ideal $\boxed{\text{of } R}$, $ra \in A$ and $ar \in A$.
Since B is an ideal $\boxed{\text{of } S}$, $sb \in B$ and $bs \in B$.
Thus, $mz = (r, s)(a, b) = (ra, sb) \in A \times B$
and $zm = (a, b)(r, s) = (ar, bs) \in A \times B$.

Thus, $A \times B$ is an ideal of $R \times S$.

4. [10 points] PICK ONE OF THE FOLLOWING:

A) Let $\phi : F \to R$ be a ring homomorphism where F is a field and R is a ring. Prove: (i) The kernel $\ker(\phi)$ is an ideal of F, and (ii) If ϕ is onto and $R \neq \{0\}$ then ϕ is an isomorphism.

B) Let R and S be commutative rings with identities 1_R and 1_S respectively. (a) If A is an ideal of R and B is an ideal of S, show that $A \times B$ is an ideal of $R \times S$. (b) Show that every ideal I of $R \times S$ has the form $I = A \times B$ where A is an ideal of R and B is an ideal of S. [Hint for b: Define $A = \{a \in R \mid (a, 0) \in I\}$ and $B = \{b \in S \mid (0, b) \in I\}$.

(b) Let I be an ideal of $R \times S$.

Let $A = \{a \in R \mid (a,0) \in I\}$ and

$B = \{b \in S \mid (0,b) \in I\}$.

$\boxed{\text{Claim: } I = A \times B}$

$\boxed{A \times B \subseteq I}$: Let $(a,b) \in A \times B$. Then $(a,0) \in I$ and $(0,b) \in I$. So, $(a,b) = (a,0) + (0,b) \in I$ since I is an ideal.

$\boxed{I \subseteq A \times B}$: Let $i = (a,b) \in I$ where $a \in R$, $b \in S$. Then since I is an ideal of $R \times S$ and $(1_R, 0) \in R \times S$ we get $(a,0) = (1_R, 0)(a,b) \in I$. Thus, $a \in A$. Similarly, $(0,b) = (0, 1_S)(a,b) \in I$. So, $b \in B$. Thus, $i = (a,b) \in A \times B$.

$\boxed{\text{Claim}}$

(next page)

Claim: A is an ideal of R

- $(0,0) \in I$ since I is an ideal, hence $0 \in A = \{a \in R \mid (a,0) \in I\}$.
- Suppose $a_1, a_2 \in A$. Then $(a_1, 0), (a_2, 0) \in I$. So, $(a_1, 0) - (a_2, 0) = (a_1 - a_2, 0)$ is in I, So, $a_1 - a_2 \in A$.
- Let $a \in A$ and $r \in R$. Then $(a,0) \in I$. Since I is an ideal of $R \times S$ we get

$$(ra,0) = \underbrace{(r,0)}_{\text{in } R\times S} \underbrace{(a,0)}_{\text{in } I} \in I.$$

And

$$(ar,0) = (a,0)(r,0) \in I,$$

Therefore by the above A is an ideal of R.

Claim: B is an ideal of S

Same proof as above claim.

Putting this all together proves part (b).