Math 5680

1/30/23

Ex: Consider the p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots$$

where $p \in \mathbb{R}$.

One can show that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges iff $p > 1$.

proof: We will prove ($\Leftarrow$).
The other direction is in HW 1.

Let

$$S_k = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{k^p}$$

be the k-th partial sum.

The sequence of partial sums

$$\underbrace{\frac{1}{1^p}}_{S_1},\ \underbrace{\frac{1}{1^p}+\frac{1}{2^p}}_{S_2},\ \underbrace{\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}}_{S_3},\ldots$$

is an increasing sequence of real numbers.

Consider the sub-sequence gotten from S_{2^k-1}.

$$S_{2^1-1} = S_1 = \frac{1}{1^p} = \left(\frac{1}{2^{p-1}}\right)^0$$

$$S_{2^2-1} = S_3 = \frac{1}{1^p} + \left(\frac{1}{2^p}+\frac{1}{3^p}\right)$$

$$\leq \frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{2^p}$$

$$= \frac{1}{1^p}+\frac{2}{2^p} = \frac{1}{1^p}+\frac{1}{2^{p-1}}$$

$$= \left(\frac{1}{2^{p-1}}\right)^0 + \left(\frac{1}{2^{p-1}}\right)^1$$

$$S_{2^3-1} = S_7 = \frac{1}{1^p} + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) + \left(\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}\right)$$

$$\leq \frac{1}{1^p} + \left(\frac{1}{2^p} + \frac{1}{2^p}\right) + \left(\frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p}\right)$$

$$= \frac{1}{1^p} + \frac{2}{2^p} + \frac{4}{4^p}$$

$$= \frac{1}{1^p} + \frac{1}{2^{p-1}} + \frac{1}{4^{p-1}}$$

$$= \left(\frac{1}{2^{p-1}}\right)^0 + \left(\frac{1}{2^{p-1}}\right)^1 + \left(\frac{1}{2^{p-1}}\right)^2$$

In general you can show that

$$S_{2^k-1} \leq \left(\frac{1}{2^{p-1}}\right)^0 + \left(\frac{1}{2^{p-1}}\right)^1 + \left(\frac{1}{2^{p-1}}\right)^2 + \dots + \left(\frac{1}{2^{p-1}}\right)^{k-1}.$$

Note that since $p>1$ we know

$$\left|\frac{1}{2^{p-1}}\right| < \frac{1}{2^0} = 1.$$

Thus, $\sum_{n=0}^{\infty} \left(\frac{1}{2^{p-1}}\right)^n$ converges.

From above we get

$$S_{2^k-1} \leq \sum_{n=0}^{\infty} \left(\frac{1}{2^{p-1}}\right)^n = \frac{1}{1 - 1/2^{p-1}}$$

There's no k here!

$$\boxed{\begin{array}{c}\displaystyle\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \\ |x|<1\end{array}}$$

Consider S_ℓ where $\ell \geq 1$.

Pick a k where $\ell \leq 2^k - 1$.

Then, since we have an increasing sequence,

$$S_\ell \leq S_{2^k-1} \leq \frac{1}{1 - 1/2^{p-1}}.$$

Thus, $(S_\ell)_{\ell=1}^{\infty}$ is an increasing, bounded from above, sequence of real numbers.

From the monotone convergence theorem from Math 4650 we know $(S_\ell)_{\ell=1}^{\infty}$ converges.

Thus, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges when $p>1$.

($\Rightarrow$) HW 1. $\blacksquare$

<u>Def:</u> Let $\sum_{n=1}^{\infty} a_n$ be a series of complex numbers. We say that $\sum_{n=1}^{\infty} a_n$ <u>converges absolutely</u> if $\sum_{n=1}^{\infty} |a_n|$ converges.

<u>Ex:</u> Consider

$$\sum_{n=1}^{\infty} \frac{i^n}{n^2} = \frac{i}{1^2} - \frac{1}{2^2} - \frac{i}{3^2} + \frac{1}{4^2} + \frac{i}{5^2} - \cdots$$

Does this series converge absolutely?

$$\sum_{n=1}^{\infty} \left| \frac{i^n}{n^2} \right| = \sum_{n=1}^{\infty} \frac{|i|^n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} \leftarrow$$ converges $p=2$ series

Yes! The series $\sum_{n=1}^{\infty} \frac{i^n}{n^2}$ converges absolutely.

Theorem: If $\sum_{n=1}^{\infty} a_n$ converges absolutely, then $\sum_{n=1}^{\infty} a_n$ converges.

proof: Let

$$S_k = \sum_{n=1}^{k} a_n = a_1 + a_2 + \dots + a_k$$

and

$$\hat{S}_k = \sum_{n=1}^{k} |a_n| = |a_1| + |a_2| + \dots + |a_k|$$

be the partial sums of $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} |a_n|$

Our assumption is that the sequence $(\hat{S}_k)_{k=1}^{\infty}$ converges.

We want to show this implies that $(S_k)_{k=1}^{\infty}$ converges.

Let $\varepsilon > 0$.

Since $(\hat{S}_k)_{k=1}^{\infty}$ converges, it is a Cauchy sequence.

Thus there exists $N > 0$ where if $n \geq m \geq N$, then $|\hat{S}_n - \hat{S}_m| < \varepsilon$

Suppose $n \geq m \geq N$.

<u>case 1:</u> If $n = m$, then

$$|S_n - S_m| = |S_n - S_n| = 0 < \varepsilon.$$

<u>case 2:</u> If $n > m$, then

$$|S_n - S_m| = \left| \sum_{k=1}^{n} a_k - \sum_{k=1}^{m} a_k \right|$$

$$\overset{n>m}{=} \left| \sum_{k=m+1}^{n} a_k \right|$$

$$\overset{\Delta}{\leq} \sum_{k=m+1}^{n} |a_k|$$

$$\overset{n>m}{=} \sum_{k=1}^{n} |a_k| - \sum_{k=1}^{m} |a_k|$$

$$\overset{n>m}{=} \left| \sum_{k=1}^{n} |a_k| - \sum_{k=1}^{m} |a_k| \right|$$

$$= |\hat{s}_n - \hat{s}_m| < \varepsilon$$

Thus, $(s_k)_{k=1}^{\infty}$ is a Cauchy sequence and hence converges.

$\boxed{\text{Caüchy}}$

Class question: $n=5, m=3$

$$\sum_{k=1}^{5} a_k - \sum_{k=1}^{3} a_k = a_1 + a_2 + a_3 + a_4 + a_5 - a_1 - a_2 - a_3$$

$$= a_4 + a_5$$

$$= \sum_{k=4}^{5} a_k$$

Consequently, since $\sum_{n=1}^{\infty} \frac{i^n}{n^2}$ converges absolutely, it also converges.

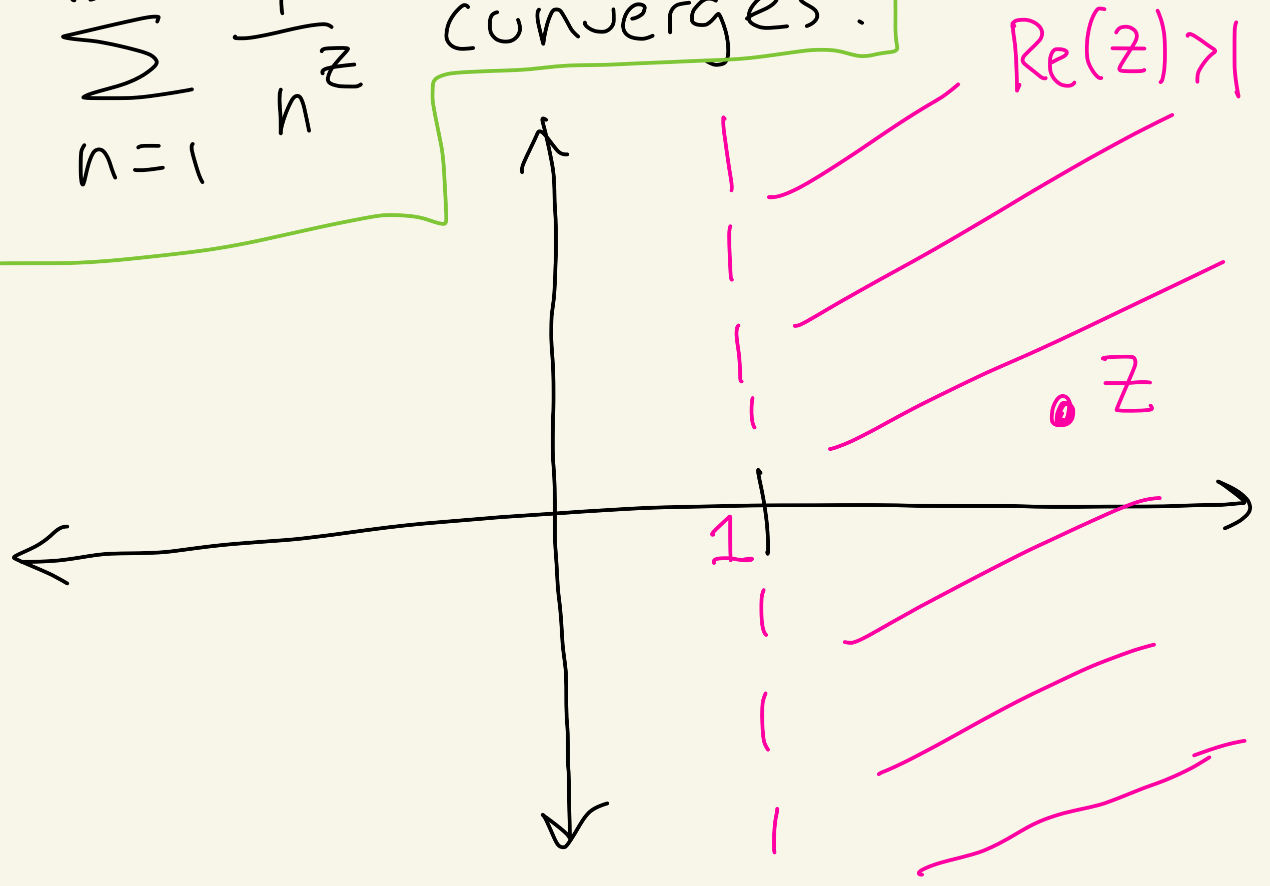
Ex: (Riemann zeta function)

Consider

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} = \frac{1}{1^z} + \frac{1}{2^z} + \frac{1}{3^z} + \cdots$$

where $z \in \mathbb{C}$.

We will show that if $\mathrm{Re}(z) > 1$ then $\sum_{n=1}^{\infty} \frac{1}{n^z}$ converges.



Recall

$$\frac{1}{n^z} = n^{-z} = e^{-z\log(n)}$$

where

$$\log(n) = \ln|n| + i\arg(n)$$

will will use the principal branch of log where

$$-\pi < \arg(n) < \pi.$$