Math 5680

2/1/23

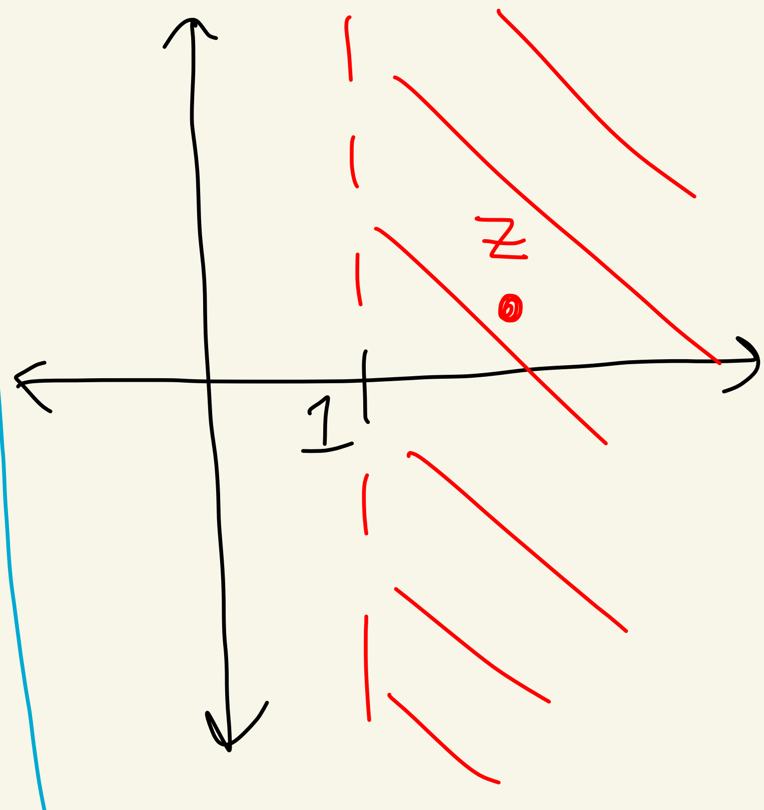
Ex: (Riemann zeta function)

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} = \frac{1}{1^z} + \frac{1}{2^z} + \frac{1}{3^z} + \cdots$$

where $z \in \mathbb{C}$

We will show that if $\mathrm{Re}(z) > 1$ then $\sum_{n=1}^{\infty} \frac{1}{n^z}$ converges.

proof: Let $z = x + iy$ with $x = \mathrm{Re}(z) > 1$.



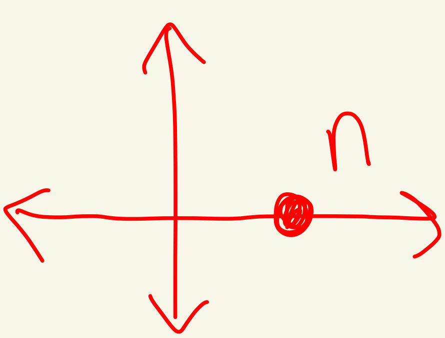

We will show the series converges absolutely given $x > 1$.

We have that

$$\left|\frac{1}{n^z}\right| = \left|n^{-z}\right| \overset{\text{def of complex power}}{=} \left|e^{-z\log(n)}\right| =$$

$$\boxed{\begin{array}{l} \log(n) = \ln|n| + i\arg(n) \\ -\pi < \arg(n) < \pi \\ \text{(principal branch of log)} \end{array}}$$

$$= \left|e^{-z[\ln(n)+i0]}\right|$$




$\arg(n) = 0$

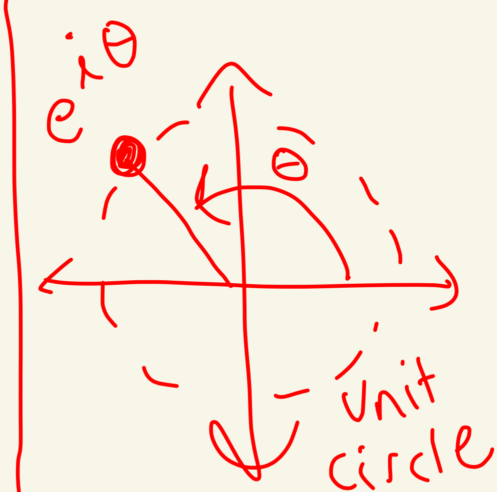
$$= \left|e^{-\overbrace{(x+iy)}^{z}\ln(n)}\right|$$

$$= \left|e^{-x\ln(n)}\, e^{-iy\ln(n)}\right|$$

$$= \left|e^{-x\ln(n)}\right|\left|e^{-iy\ln(n)}\right|$$

If $\theta \in \mathbb{R}$

$\left|e^{i\theta}\right| = 1$




$$= \left|e^{-x\ln(n)}\right|$$

these are all real #s!

$$= e^{-x\ln(n)}$$

$$= e^{\ln(n^{-x})} = n^{-x} = \frac{1}{n^x}$$

Thus,

$$\sum_{n=1}^{\infty}\left|\frac{1}{n^z}\right| = \sum_{n=1}^{\infty}\frac{1}{n^x}$$

which converges since this is a p-series with $p = x > 1$.

Thus, $\sum_{n=1}^{\infty} \frac{1}{n^z}$ converges absolutely when $Re(z) > 1$ and so it converges when $Re(z) > 1$.

$\blacksquare$

# TOPIC 2 – Sequences and series of functions

<u>Def:</u> Suppose that $A \subseteq \mathbb{C}$ and $f_n : A \to \mathbb{C}$ for each $n = 1, 2, 3, 4, \ldots$

[$A$ is a common domain of $f_1, f_2, f_3, \ldots$]

Let $f : A \to \mathbb{C}$. We say that

① $(f_n)_{n=1}^{\infty}$ <u>converges pointwise to $f$ on $A$</u> if for each $z \in A$ we know $\lim_{n\to\infty} f_n(z) = f(z)$

① says: Given $z \in A$ and $\varepsilon > 0$

there exists $N>0$ where if $n \geq N$ then $|f_n(z) - f(z)| < \varepsilon$

N depends on $z$ and $\varepsilon$

(2) Let $f: A \to \mathbb{C}$. We say that $(f_n)_{n=1}^{\infty}$ <u>converges uniformly to $f$ on $A$</u> if for every $\varepsilon > 0$ there is an $N > 0$ where if $n \geq N$, then $|f_n(z) - f(z)| < \varepsilon$ for all $z \in A$.

N depends on $\varepsilon$ and works for all $z$

③ A series $\sum_{n=1}^{\infty} g_n(z)$ is said

to <u>converge pointwise</u> if the

corresponding partial sums

$$\underbrace{S_k(z)}_{\text{taking the place of } f_n} = \sum_{n=1}^{k} g_n(z) = g_1(z) + g_2(z) + \dots + g_k(z)$$

converge pointwise.

A series $\sum_{n=1}^{\infty} g_n(z)$ is said to

<u>converge uniformly</u> if the corresponding

partial sums $S_k(z) = \sum_{n=1}^{k} g_n(z)$

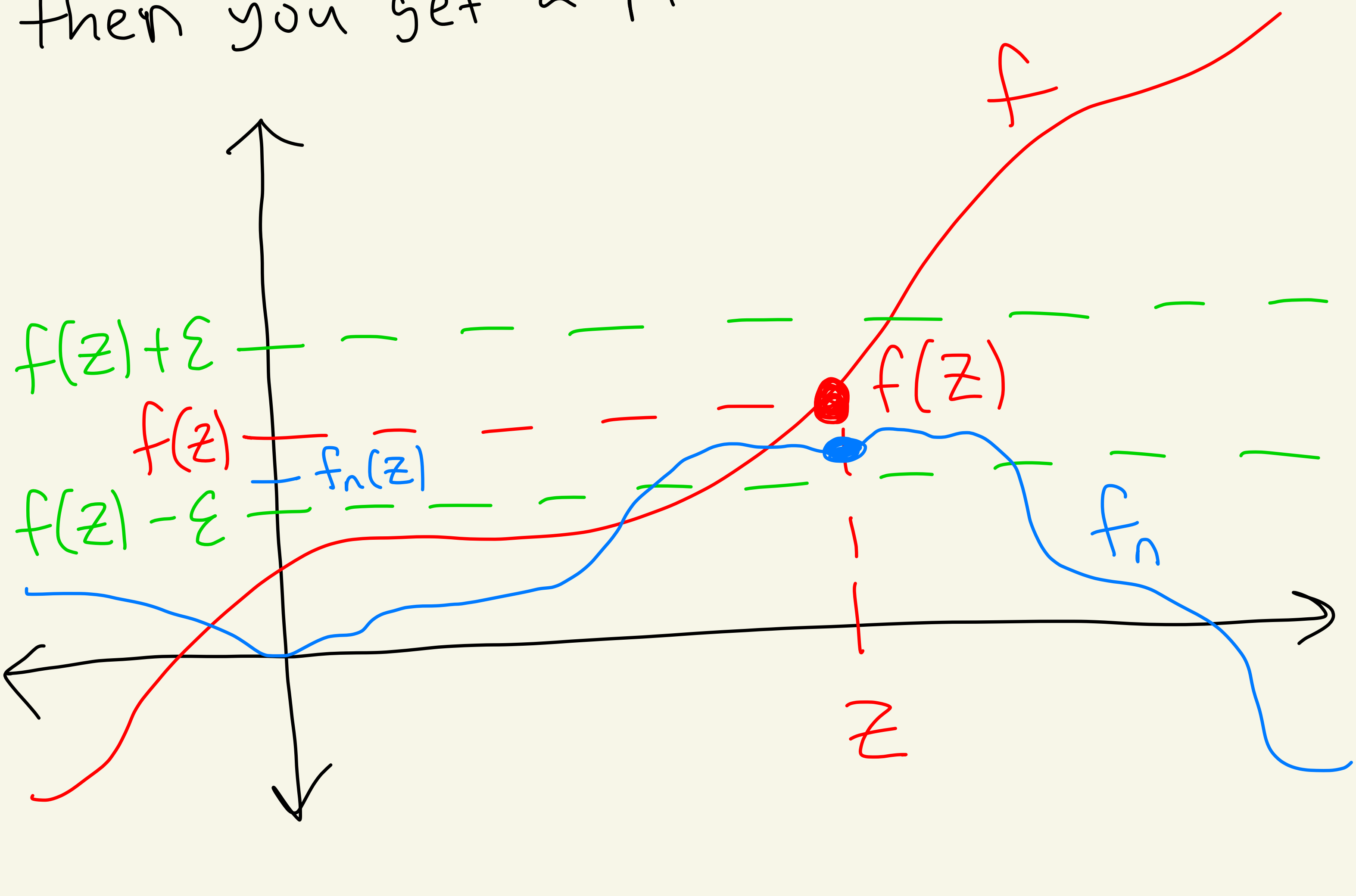
converge uniformly.

# PICTURE OF $f_n \to f$ pointwise in the real numbers

$f_n : \mathbb{R} \to \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$, $A = \mathbb{R}$

Fix $z \in \mathbb{R}$ and $\varepsilon > 0$.

There will exist $N > 0$ where if $n \geq N$ then you get a picture like this:

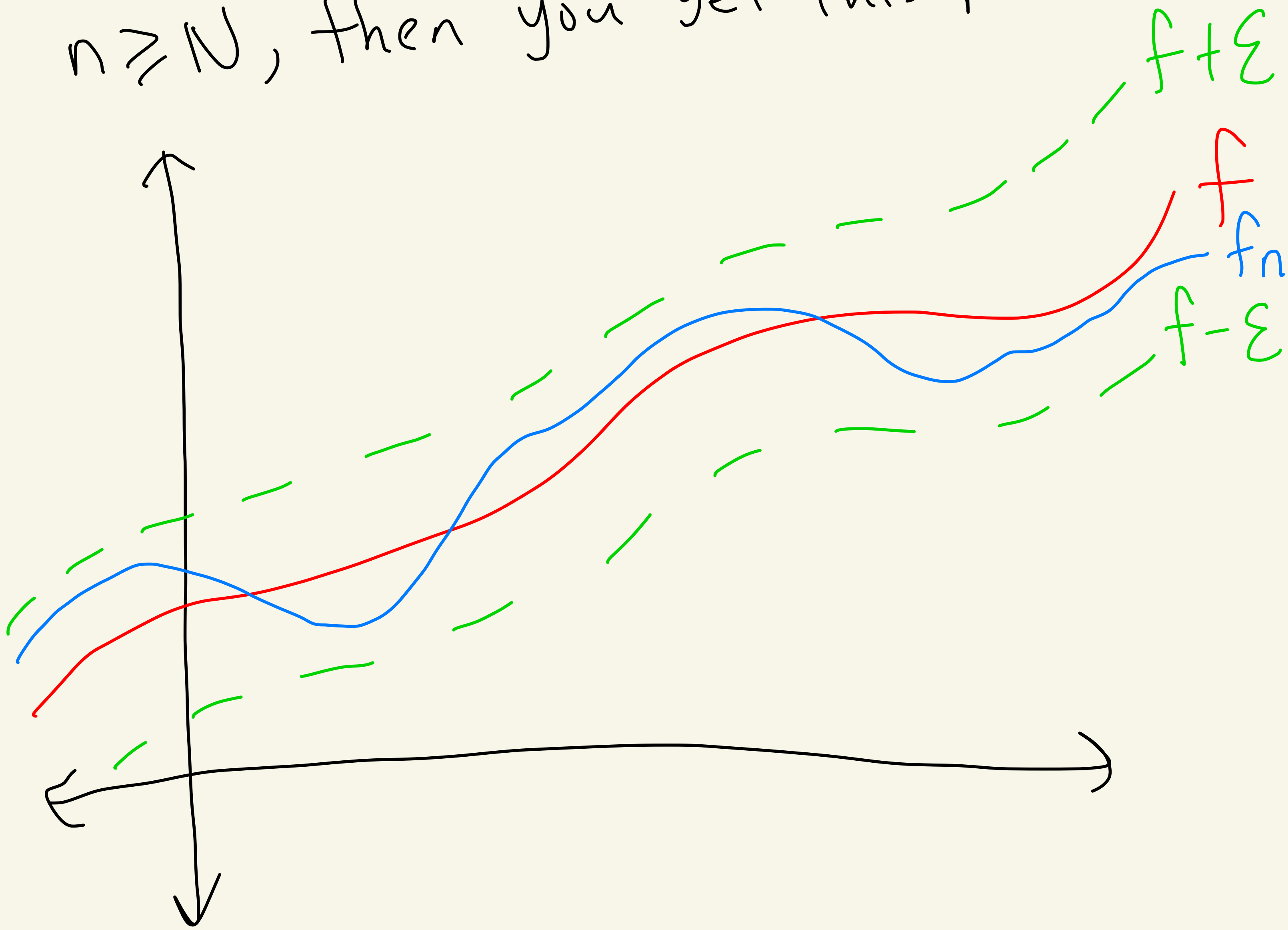


# Picture for uniform convergence in $\mathbb{R}$

$f_n : \mathbb{R} \to \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$, $A = \mathbb{R}$

Fix $\varepsilon > 0$.

There will exist $N > 0$ where if $n \geq N$, then you get this picture



**Ex:** Let $f_n(z) = z^n$, $n \geq 1$

sequence:

$$z,\ z^2,\ z^3,\ z^4,\ z^5,\ \ldots$$

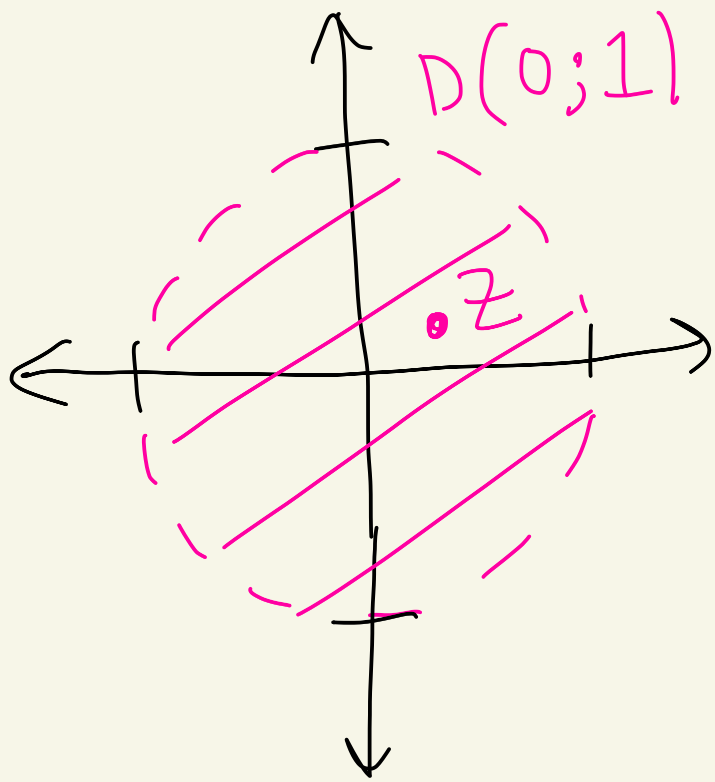
$$f_1,\ f_2,\ f_3,\ f_4,\ f_5,\ \ldots$$

Let $A = D(0;1)$.

$D(z_0;r) = \{z \mid |z - z_0| < r$

$z_0$ = center $\quad r$ = radius

Define $f: A \to \mathbb{C}$ as $f(z) = 0$ for all $z \in A$.




Then, $f_n \to f$ pointwise on $A$.

# WHY?

Riemann

Let $z \in A$. Then, $|z| < 1$.

So,

$$\lim_{n\to\infty} f_n(z) = \lim_{n\to\infty} z^n = 0 = f(z)$$

$$\lim_{n\to\infty} |z^n| = \lim_{n\to\infty} |z|^n = 0$$

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By thm in class this implies $\lim_{n\to\infty} z^n = 0$

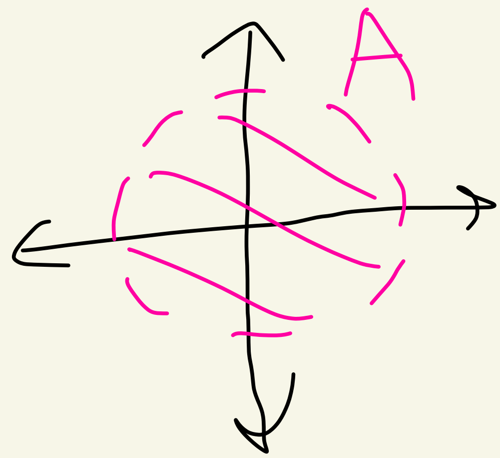
Very nice!

$\blacksquare$

**Ex:** Let $f_n(z) = \frac{z}{n}$, $n \geq 1$

sequence is:

$$z, \frac{z}{2}, \frac{z}{3}, \frac{z}{4}, \dots$$




Let $A = D(0;1)$

Let $f(z) = 0 \quad \forall z \in A$

$$\boxed{\text{Claim: } f_n \to f \text{ uniformly on } A}$$

$\to$ means converges to

**proof:** Let $\varepsilon > 0$.

Set $N > \frac{1}{\varepsilon}$.

If $n \geq N$ and $z \in A$, then

$$|f_n(z) - f(z)| = \left|\frac{z}{n} - 0\right| = \left|\frac{z}{n}\right|$$

$$= \frac{|z|}{n} < \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

$\boxed{z \in A \quad |z| < 1}$