Math 5680
2/8/23

[Super duper Δ-inequality]

<u>Theorem:</u> If $\sum_{k=1}^{\infty} a_k$ converges absolutely, then

$$\left|\sum_{k=1}^{\infty} a_k\right| \leq \sum_{k=1}^{\infty} |a_k|$$

<u>Proof:</u>

Let $s_n = \sum_{k=1}^{n} a_k$ and $\hat{s}_n = \sum_{k=1}^{n} |a_k|$. Since $\sum_{k=1}^{\infty} a_k$ converges absolutely, we know both $\lim_{n\to\infty} s_n = \sum_{k=1}^{\infty} a_k$ and

$$\lim_{n\to\infty} \hat{s}_n = \sum_{k=1}^{\infty} |a_k| \text{ both exist.}$$

We know

$$|s_n| = \left| \sum_{k=1}^{n} a_k \right| \overset{\triangle}{\leq} \sum_{k=1}^{n} |a_k| = \hat{s}_n$$

By 4650, we have

$$\lim_{n\to\infty} |s_n| \leq \lim \hat{s}_n.$$

That is,

$$\lim_{n\to\infty} \left| \sum_{k=1}^{n} a_k \right| \leq \lim_{n\to\infty} \sum_{k=1}^{n} |a_k|$$

From Math 4680,

$$\lim_{n\to\infty}\left|\sum_{k=1}^{n} a_k\right| = \left|\lim_{n\to\infty}\sum_{k=1}^{n} a_k\right|$$

4680

you can push a limit inside a continuous function

here $f(z)=|z|$ is our continuous function

So, $$\left|\lim_{n\to\infty}\sum_{k=1}^{n} a_k\right| \le \lim_{n\to\infty}\sum_{k=1}^{n} |a_k|$$

Hence
$$\left|\sum_{k=1}^{\infty} a_k\right| \le \sum_{k=1}^{\infty} |a_k|$$

$\blacksquare$

Theorem (Weierstrass M-Test)

Let $A \subseteq \mathbb{C}$.

Let $g_k : A \to \mathbb{C}$ for $k \geq 1$.

Suppose there are real constants $M_k \geq 0$ for $k \geq 1$, where

(i) $|g_k(z)| \leq M_k$ for all $z \in A$

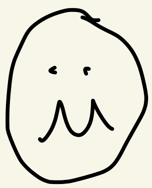
and (ii) $\sum_{k=1}^{\infty} M_k$ converges.

Then, $\sum_{k=1}^{\infty} g_k(z)$ converges absolutely and uniformly on A.

<u>Proof:</u>

Let $\hat{s}_n(z) = \sum_{k=1}^{n} |g_k(z)|$

and $t_n = \sum_{k=1}^{n} M_k$.

By  we know $\lim_{n\to\infty} t_n = \sum_{k=1}^{\infty} M_k$ exists.

Weierstrass

Let's first show $\sum_{k=1}^{\infty} g_k(z)$ converges absolutely for all $z \in A$.

To do this let's show $\hat{s}_n(z)$ is a Cauchy sequence for all $z \in A$.

This will imply $\lim_{n\to\infty} \hat{s}_n(z) = \sum_{k=1}^{\infty} |g_k(z)|$ exists $\forall z \in A$.

Let $\varepsilon > 0$.

Since, by , $\lim_{n \to \infty} t_n = \sum_{k=1}^{\infty} M_k$

converges, so (t_n) is a Cauchy sequence.

Thus, there exists $N > 0$ where if $n > m \geq N$ then $|t_n - t_m| < \varepsilon$

So, if $n > m \geq N$, then

$$\sum_{k=m+1}^{n} M_k = \sum_{k=1}^{n} M_k - \sum_{k=1}^{m} M_k$$

$$= \left| \sum_{k=1}^{n} M_k - \sum_{k=1}^{m} M_k \right|$$

$$= |t_n - t_m| < \varepsilon$$

Hence, if $n > m \geq N$ and $z \in A$, then

$$|\hat{S}_n(z) - \hat{S}_m(z)| = \left| \sum_{k=1}^{n} |g_k(z)| - \sum_{k=1}^{m} |g_k(z)| \right|$$

$$= \sum_{k=m+1}^{n} |g_k(z)| \overset{i}{\leq} \sum_{k=m+1}^{n} M_k < \varepsilon$$

Thus, $(\hat{S}_n(z))$ is a Cauchy sequence for all $z \in A$ and hence has a limit for all $z \in A$.

So, $\sum_{n=1}^{\infty} g_k(z)$ converges absolutely $\forall z \in A$.

Now it's time for UNIFORM CONVERGENCE!!!

For each $z \in A$, let

$$s_n(z) = \sum_{k=1}^{n} g_k(z).$$

We know that $\lim_{n\to\infty} s_n(z)$ exists

for each $z \in A$ from the above.

Let $s(z) = \lim_{n\to\infty} s_n(z) = \sum_{k=1}^{\infty} g_k(z).$

If $z \in A$, then

$$|s(z) - s_n(z)| = \left| \sum_{k=1}^{\infty} g_k(z) - \sum_{k=1}^{n} g_k(z) \right|$$

$$= \left| \sum_{k=n+1}^{\infty} g_k(z) \right|$$

Check out HW 1 #2 solution

$$\leq \sum_{k=n+1}^{\infty} |g_k(z)|$$

Super-duper! Δ

$$\leq \sum_{k=n+1}^{\infty} M_k$$

Let $\varepsilon > 0$.

Since $\sum_{k=1}^{\infty} M_k$ converges, there exists $N > 0$ where if $n \geq N$ then

$$\Big| \underbrace{\sum_{k=1}^{\infty} M_n}_{\text{limit}} - \underbrace{\sum_{k=1}^{n} M_n}_{\text{partial sum}} \Big| < \varepsilon$$

So if $n \geq N$, then

$$\sum_{k=n+1}^{\infty} M_k = \sum_{k=1}^{\infty} M_k - \sum_{k=1}^{n} M_k$$

$$= \Big| \sum_{k=1}^{\infty} M_k - \sum_{k=1}^{n} M_k \Big| < \varepsilon$$

Therefore, hence, ergo, we have that if $z \in A$ and $n \geq N$ then

$$|s(z) - s_n(z)| \leq \sum_{k=n+1}^{\infty} M_k < \varepsilon$$

Consequently, $s_n \longrightarrow s$ uniformly on A.

That is, $\sum_{k=1}^{\infty} g_k(z)$ converges uniformly on A.

Weierstrass