Math 5680
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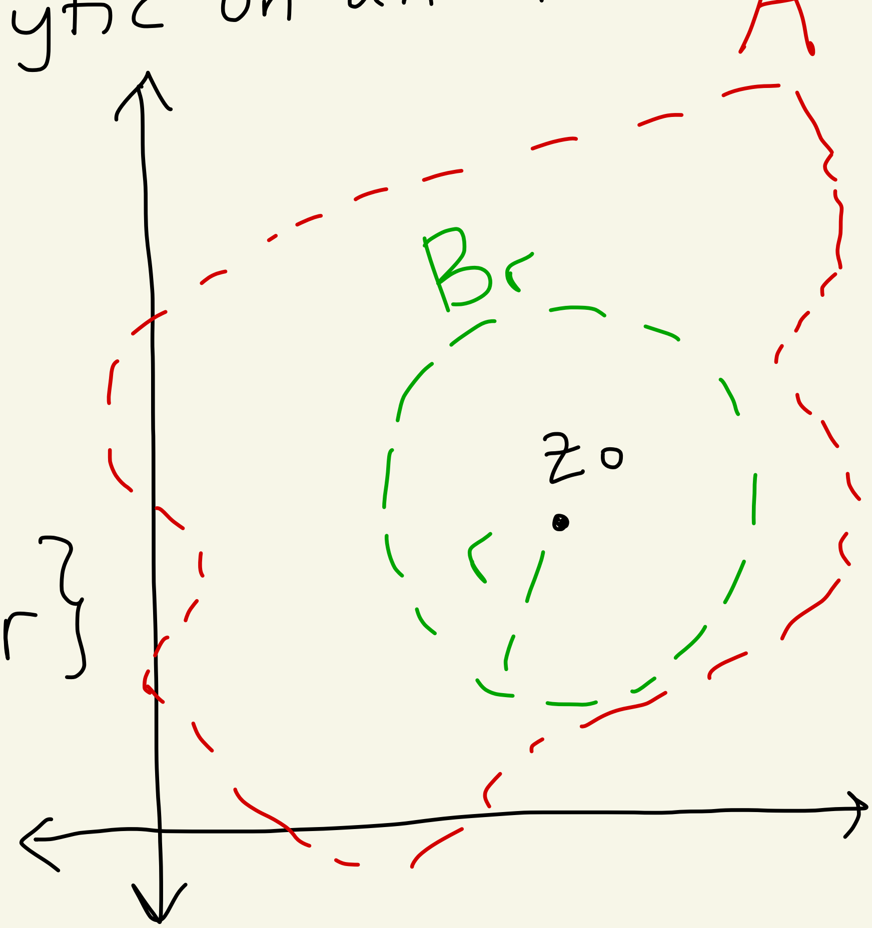
Theorem (Taylor's Theorem)

Let f be analytic on an open set $A \subseteq \mathbb{C}$.



Let $z_0 \in A$.

Let

$$B_r = \{z \mid |z - z_0| < r\}$$

$$= D(z_0; r)$$

Suppose $B_r \subseteq A$.

Then the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$$

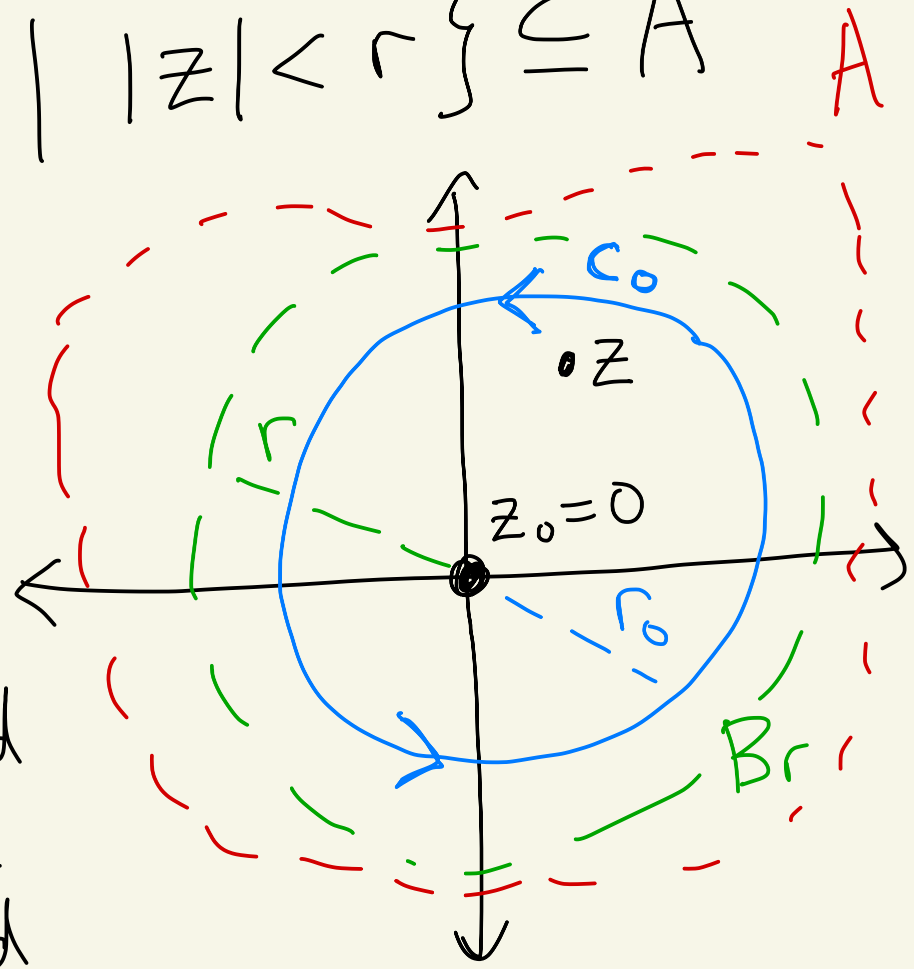
Taylor series for f centered at z_0

converges to $f(z)$ for all $z \in B_r$

proof; We first prove the theorem when $z_0 = 0$.

Let $B_r = \{ z \mid |z| < r \} \subseteq A$

Let $z \in B_r$.



Let C_0 denote some circle of radius r_0, centered at $z_0 = 0$, oriented counter-clockwise that is contained inside the disc B_r but is large enough so that z is interior to C_0.

Since f is analytic inside and on C_0 and z is interior to C_0 we have

$$f(z) = \frac{1}{2\pi i} \int_{C_0} \frac{f(\varsigma)}{\varsigma - z} \, d\varsigma$$

$\boxed{\text{Cauchy integral theorem}}$

Recall that when $w \neq 1$ then

$$\sum_{n=0}^{N-1} w^n = 1 + w + w^2 + \cdots + w^{N-1} = \frac{1-w^N}{1-w}$$

$$= \frac{1}{1-w} - \frac{w^N}{1-w}$$

Thus, if $w \neq 1$, then

$$\frac{1}{1-w} = \sum_{n=0}^{N-1} w^n + \frac{w^N}{1-w}$$

for $N \geq 1$.

$\frac{z}{\rho} \neq 1$ in the integral because ρ is on C_0 and z is inside C_0

Hence,

$$\frac{1}{\rho - z} = \left(\frac{1}{\rho}\right)\left(\frac{1}{1-\left(\frac{z}{\rho}\right)}\right)$$

$$= \left(\frac{1}{\rho}\right)\left(\left(\sum_{n=0}^{N-1}\left(\frac{z}{\rho}\right)^n\right) + \frac{\left(\frac{z}{\rho}\right)^N}{1-\frac{z}{\rho}}\right)$$

$$= \left(\sum_{n=0}^{N-1}\left(\frac{1}{\rho^{n+1}}\right) z^n\right) + z^N \frac{1}{(\rho - z)\rho^N}$$

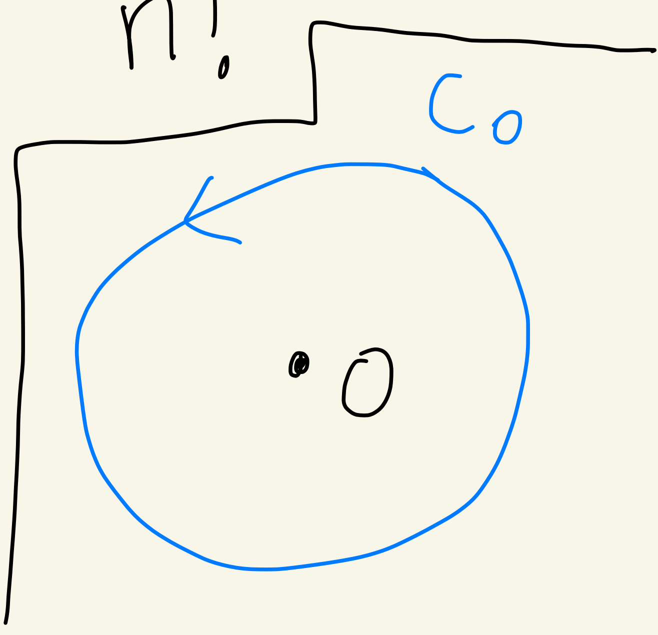
Thus,
$$f(z) = \frac{1}{2\pi i}\int_{C_0} \frac{f(\rho)}{\rho - z}\,d\rho$$

$$= \frac{1}{2\pi i}\sum_{n=0}^{N-1}\left(\int_{C_0}\frac{f(\rho)}{\rho^{n+1}}\,d\rho\right) z^n$$

$$+ \frac{z^N}{2\pi i}\int_{C_0}\frac{f(\rho)}{(\rho - z)\rho^N}\,d\rho$$

By Cauchy's integral theorem, since f is analytic in and on C_0 and 0 is interior to C_0 we have that

$$\underbrace{\frac{1}{2\pi i}\int_{C_0}\frac{f(\rho)}{\rho^{n+1}}\,d\rho}_{\frac{1}{2\pi i}\int_{C_0}\frac{f(\rho)}{(\rho-0)^{n+1}}\,d\rho} = \frac{f^{(n)}(0)}{n!}$$



Thus,

$$\boxed{f(z) = \sum_{n=0}^{N-1} \frac{f^{(n)}(0)}{n!} z^n + \rho_N(z)}$$

where

$$\boxed{\rho_N(z) = \frac{z^N}{2\pi i} \int_{C_0} \frac{f(\zeta)}{(\zeta - z)\zeta^n} d\zeta}$$

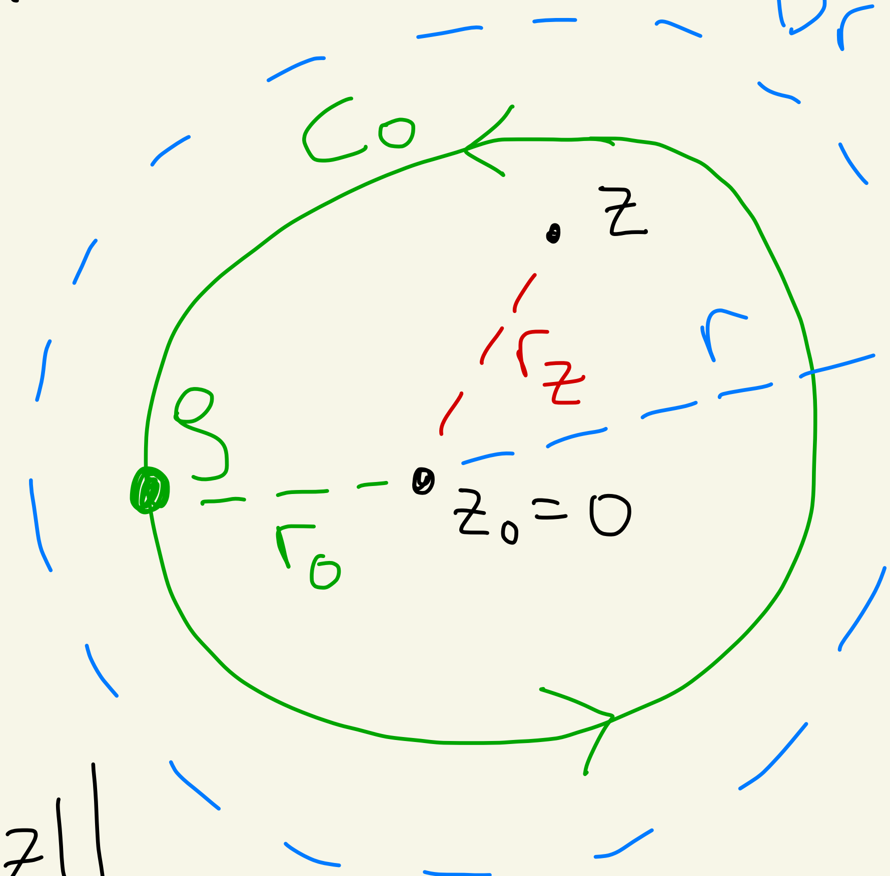
We will show that $\rho_N(z) \to 0$ as $N \to \infty$.

This will imply that

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$$

which will complete the proof of the $z_0 = 0$ case.

Let's show $\rho_N(z) \to 0$ as $N \to \infty$.



Let $r_z = |z|$.

If ρ is on C_0, then

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$$|\rho - z| \geq \big||\rho| - |z|\big|$$

$$= |r_0 - r_z| = r_0 - r_z$$

$r_0 > r_z$

$r_0 - r_z > 0$

By the max-modulus theorem from 4680 or from topology (since f is continuous on the compact set C_0)

there exists $M > 0$ where

$$|f(\rho)| \leq M$$

for all ρ on C_0.

Thus,

$$|\rho_N(z)| = \left| \frac{z^N}{2\pi i} \int_{C_0} \frac{f(\rho)}{(\rho - z)\rho^N} \, d\rho \right|$$

$|i| = 1$

$$= \frac{|z|^N}{2\pi} \left| \int_{C_0} \frac{f(\rho)}{(\rho - z)\rho^N} \, d\rho \right|$$

$|z| = r^z$

$|f(\rho)| \leq M$

$\frac{1}{|\rho - z|} \leq \frac{1}{r_0 - r_z}$

$|\rho^N| = r_0^N$

$$\leq \frac{r_z^N}{2\pi} \cdot \frac{M}{(r_0 - r_z) r_0^N} \cdot \underbrace{2\pi r_0}_{\text{arclength of } C_0}$$

$$= \left(\frac{M r_0}{r_0 - r_z} \right) \left(\frac{r_z}{r_0} \right)^N \to 0$$

$$\boxed{\left|\frac{f(\rho)}{(\rho - z)\rho^N}\right| \le \frac{M}{(r_0 - r_z)r_0^N}}$$ as $N \to \infty$

because $0 < \frac{r_z}{r_0} < 1$.

Thus, $\rho_N(z) \to 0$ as $N \to \infty$

So, $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$.

This concludes $z_0 = 0$ case.

Now we prove the general case.

Let z_0 be arbitrary.

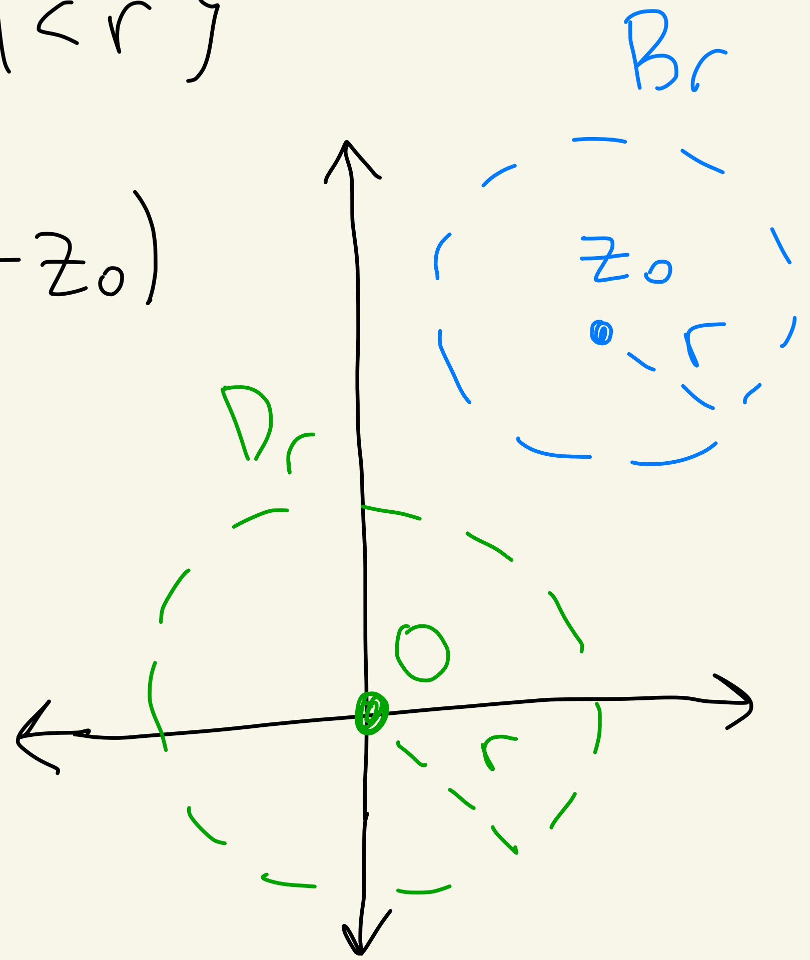
Suppose f is analytic on

$$B_r = \{ z \mid |z - z_0| < r \}$$

Let $g(z) = f(z+z_0)$

Then g is analytic on

$$D_r = \{ z \mid |z| < r \}$$



Thus by the previous case we know that

$$g(z) = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} z^n \quad \text{for all } z \in D_r$$

Then

$$f(z+z_0) = g(z) = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} z^n$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} z^n \quad \forall z \in D_r$$

$\boxed{g(z) = f(z+z_0)}$

$$g^{(n)}(z) = f^{(n)}(z+z_0)$$

$$g^{(n)}(0) = f^{(n)}(z_0)$$

Plug $z-z_0$ in for z to get that

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n \quad \forall z \in B_r$$

$\blacksquare$