Math 5680
3/13/23

$\boxed{\text{HW 2}}$ 6

Suppose that $\sum_{k=1}^{\infty} g_k(z)$ converges uniformly on $A \subseteq \mathbb{C}$. Prove that the sequence $(g_k)_{k=1}^{\infty}$ converges uniformly to the zero function f_0 on A.

$[f_0 : A \to \mathbb{C},\ f(z) = 0\ \forall z \in A]$

<u>proof:</u> Let $\varepsilon > 0$. Let $S(z) = \sum_{k=1}^{\infty} g_k(z)$.

Let $S_n(z) = \sum_{k=1}^{n} g_k(z)$ be the n-th partial sum of $\sum_{k=1}^{\infty} g_k(z)$.

Since $\sum_{k=1}^{\infty} g_k(z)$ converges uniformly on A, there exists $N > 0$ where if $n \geq N$ then $|S_n(z) - S(z)| < \varepsilon/2$ for all $z \in A$.

Thus, if $n \geq N+1$ and $z \in A$, then

$$|g_n(z) - \underbrace{0}_{f_0(z)}| = |g_n(z)|$$

$$= \left| \sum_{k=1}^{n} g_k(z) - \sum_{k=1}^{n-1} g_k(z) \right|$$

$$= |S_n(z) - S_{n-1}(z)|$$

$$
\begin{aligned}
&= \left| S_n(z) - S(z) + S(z) - S_{n-1}(z) \right| \\
&\leq \left| S_n(z) - S(z) \right| + \left| S(z) - S_{n-1}(z) \right| \\
&= \left| S_n(z) - S(z) \right| + \left| S_{n-1}(z) - S(z) \right| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}
$$

$n \geq N+1 \Rightarrow$

$n-1 \geq N$

$n \geq N$

Thus, $(g_n)_{n=1}^{\infty}$ converges uniformly to the zero function on A.

$\boxed{\text{HW 1 - \#4}}$ Let $\sum_{k=1}^{\infty} a_k$ be a series of complex numbers.

__Prove:__ $\sum_{k=1}^{\infty} a_k$ converges iff for every $\varepsilon > 0$ there exists $N > 0$ such that if $n \geq N$ then $\left| \sum_{k=n+1}^{n+p} a_k \right| < \varepsilon$ for $p = 1, 2, 3, \ldots$

__Proof:__ Let $S_n = \sum_{k=1}^{n} a_k$ be the n-th partial sum.

($\Rightarrow$) Suppose $\sum_{k=1}^{\infty} a_k$ converges.

Thus, $(S_n)_{n=1}^{\infty}$ is a convergent sequence, and thus is a Cauchy sequence.

Let $\varepsilon > 0$.

Then there exists $N > 0$ where if $n, m \geq N$ then $|S_m - S_n| < \varepsilon$.

Thus, if $n \geq N$ and $p \geq 1$, then

$$\left| \sum_{k=n+1}^{n+p} a_k \right| = \left| \sum_{k=1}^{n+p} a_k - \sum_{k=1}^{n} a_k \right|$$

$= |s_{n+p} - s_n|$

since $n \geq N$
$m = n+p \geq N$

$< \varepsilon$

($\Leftarrow$) Let $\varepsilon > 0$.

We are assuming that there exists $N > 0$ where if $q \geq N$ then $\left|\sum_{k=q+1}^{q+p} a_k\right| < \varepsilon$ for $p \geq 1$.

Suppose $m, n \geq N$.

WLOG assume $m \geq n$.

<u>Case 1:</u> Suppose $m = n$.
Then, $|s_m - s_n| = |s_n - s_n| = 0 < \varepsilon$

<u>Case 2:</u> Suppose $m > n$.

Then $m = n+p$ where $p \geq 1$.

And,

$$
\begin{aligned}
|s_m - s_n| &= |s_{n+p} - s_n| \\
&= \left| \sum_{k=1}^{n+p} a_k - \sum_{k=1}^{n} a_k \right| \\
&= \left| \sum_{k=n+1}^{n+p} a_k \right| < \varepsilon
\end{aligned}
$$

plug n in for 7

Thus, if $m, n \geq N$, then

$$|s_m - s_n| < \varepsilon.$$

So, (s_n) is a Cauchy sequence and hence converges.

Thus, $\sum_{k=1}^{\infty} a_k$ converges. $\blacksquare$

$\boxed{\text{HW 2 - (1)(b)}}$

Let $A = \mathbb{R} \subseteq \mathbb{C}$.

$f_n : A \to \mathbb{C}$ for $n \geq 2$ for

$$f_n(x) = \begin{cases} -1 & \text{for } x \leq -1/n \\ nx & \text{for } -1/n < x < 1/n \\ 1 & \text{for } 1/n \leq x \end{cases}$$

Define $f : A \to \mathbb{C}$ by

$$f(x) = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$$

Prove $f_n \to f$ pointwise on $A = \mathbb{R}$.

proof: Let $x \in A = \mathbb{R}$.

We need to show that

$$\lim_{n \to \infty} f_n(x) = f(x).$$

Case 1: Suppose $x = 0$. Then,

$$\lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} f_n(0) = \lim_{n \to \infty} n(0)$$

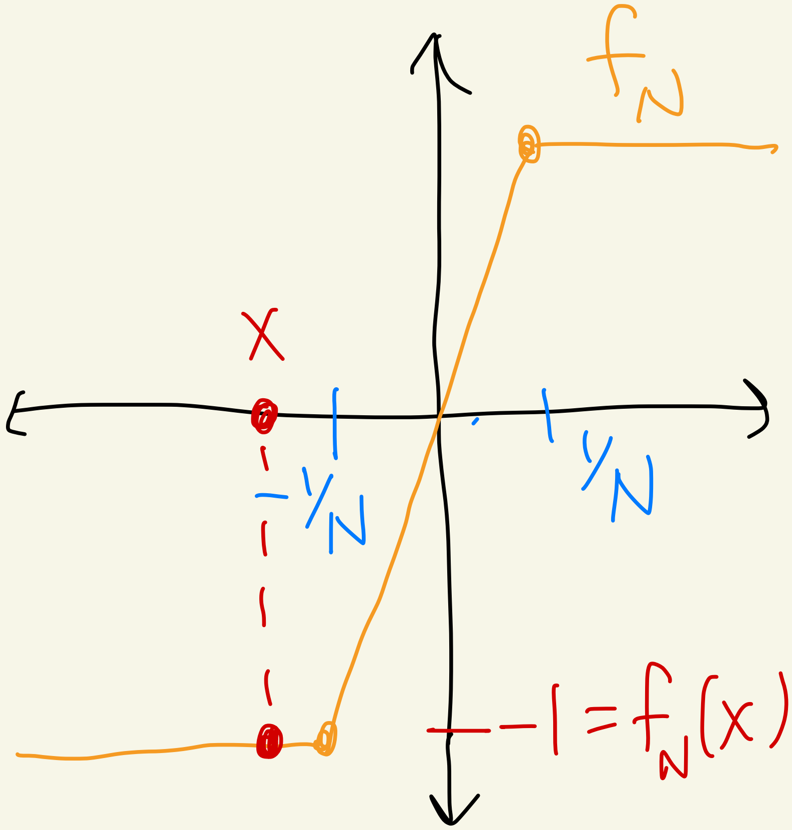
$$= 0$$

$$= f(0)$$

$$= f(x).$$

Case 2: Suppose $x < 0$.

Since $-\frac{1}{N} \to 0$ as $N \to \infty$ there exists $N > 0$ where $x < -\frac{1}{N} < 0$.



Then, if $n \geq N$ we have that

$$x < -\frac{1}{N} \leq -\frac{1}{n} \text{ and so } f_n(x) = -1.$$

Let $\varepsilon > 0$.

So, if $n \geq N$, then

$$|f_n(x) - f(x)| = |-1 - (-1)|$$

$$= 0 < \varepsilon.$$

Thus, $\lim_{n \to \infty} f_n(x) = f(x)$.

<u>Case 3:</u> Suppose $0 < x$.

Same idea as case 2.

See solutions. $\blacksquare$