Math 5680
3/8/23

Today - Continue power series (topic 3)

Monday - back at school

review for test

new stuff if time

Weds - Test 1

Ex: Let's find the radius of convergence of

$$\sum_{n=0}^{\infty} \boxed{\frac{(-1)^{n-1}}{\pi^n} z^n} = -1 + \frac{1}{\pi} z - \frac{1}{\pi^2} z^2 + \frac{1}{\pi^3} z^3 + \cdots$$

$\leftarrow \boxed{b_k}$

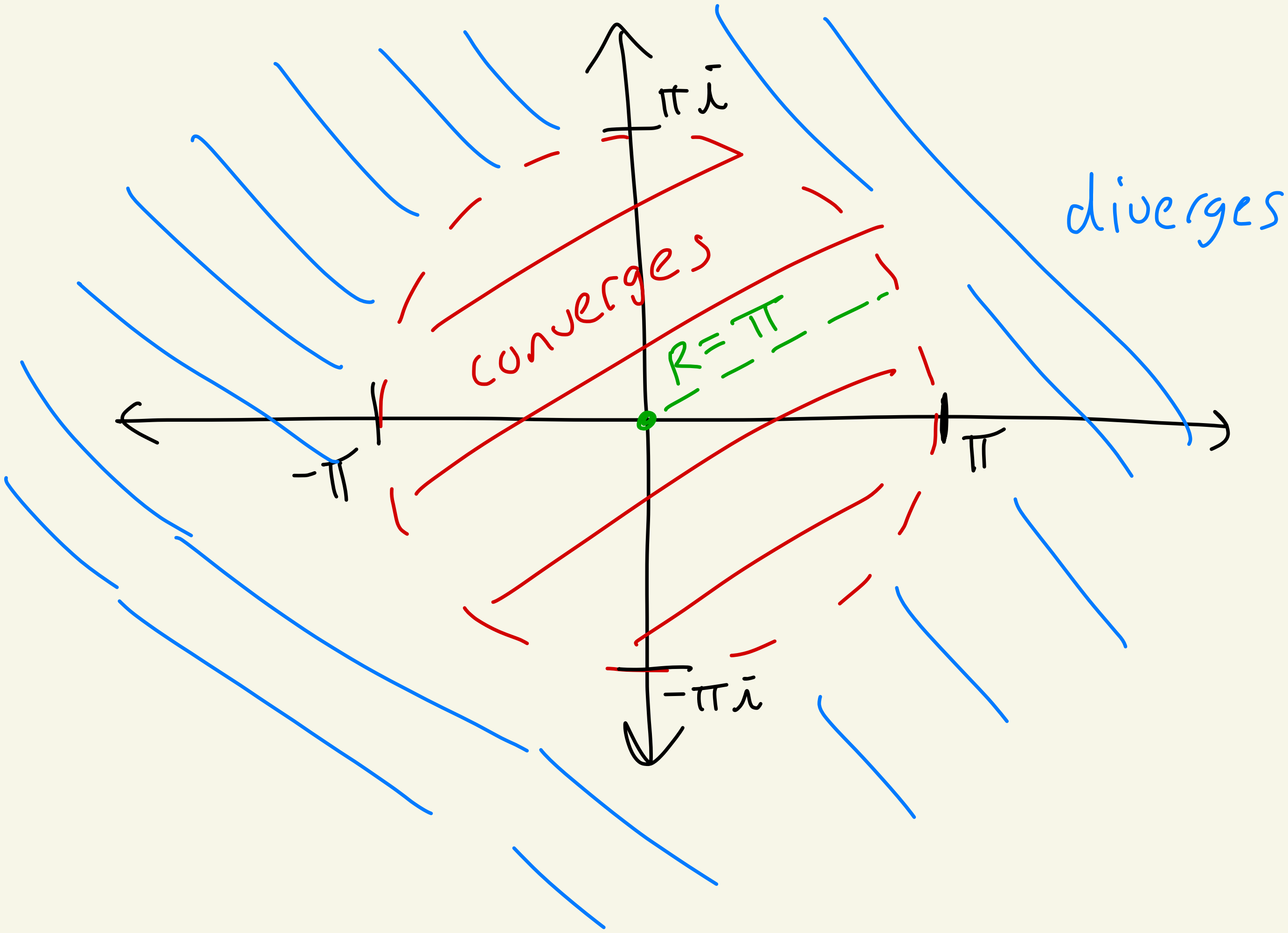
Use the ratio test

$$\lim_{k \to \infty} \left| \frac{b_{k+1}}{b_k} \right| = \lim_{k \to \infty} \left| \underbrace{\frac{(-1)^k}{\pi^{k+1}} z^{k+1}}_{b_{k+1}} \cdot \underbrace{\frac{\pi^k}{(-1)^{k-1}} \cdot \frac{1}{z^k}}_{\frac{1}{b_k}} \right|$$

$$= \lim_{k \to \infty} \left| \frac{z}{\pi} \right| = \frac{|z|}{\pi}$$

The ratio test says the series will converge if $\frac{|z|}{\pi} < 1$ and diverge if $\frac{|z|}{\pi} > 1$.

We get convergence if $|z| < \pi$ and divergence if $|z| > \pi$.



Unknown on boundary of circle.

radius of convergence is $R = \pi$

Let's now use Taylor's theorem.

Ex: $f(z) = e^z$

Let's find the Taylor series for $f(z) = e^z$ at $z_0 = 0$.

$f(z) = e^z$
$f'(z) = e^z$
$f''(z) = e^z$
$\vdots \qquad \vdots$
$f^{(k)}(z) = e^z$

Taylor series is

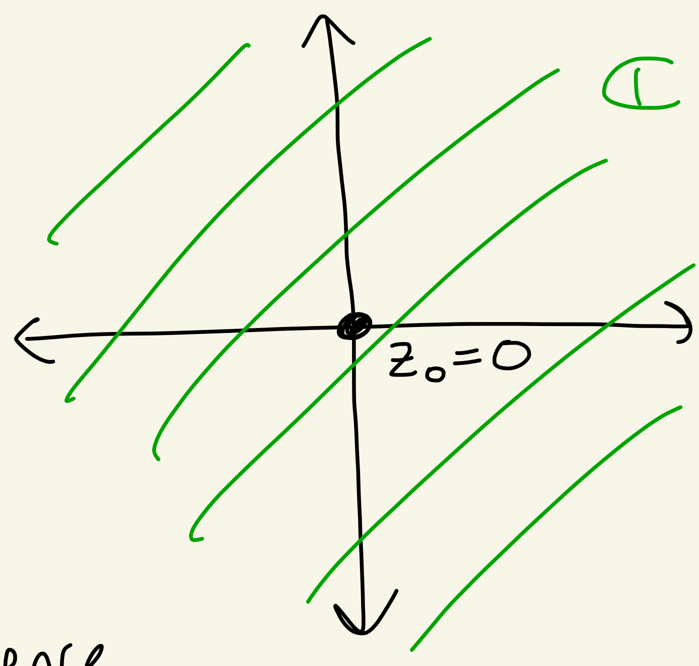
$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}(z-0)^k = \sum_{k=0}^{\infty} \frac{1}{k!} z^k$$

$f(z) = e^z$ is analytic on all of $\mathbb{C}$.

By Taylor's theorem

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}$$

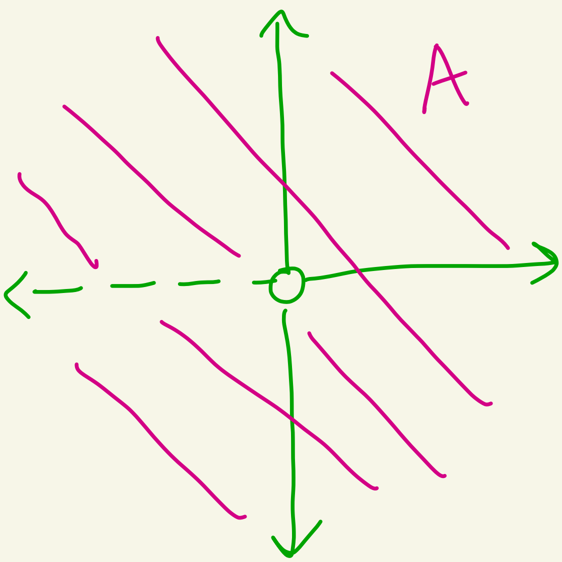
for all $z \in \mathbb{C}$.



The radius of convergence is $R = \infty$

<u>Ex:</u> Let $f(z) = \log(1+z)$ where we are using the principal branch of log.

4680 Recap



$$\log(w) = \ln|w| + i\arg(w)$$

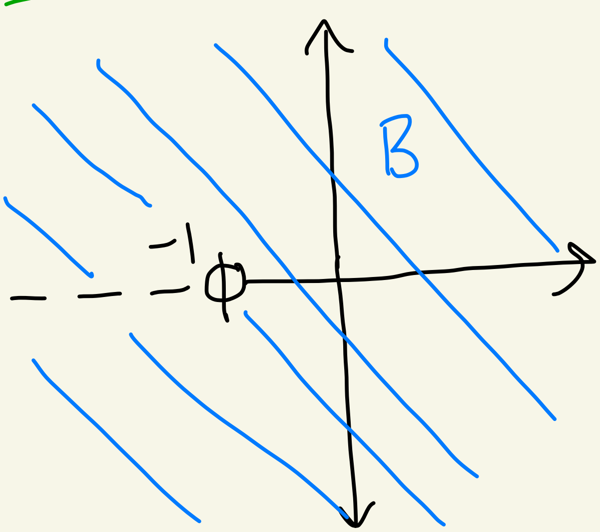
$$-\pi < \arg(w) < \pi$$

(principal branch)

This branch of log is analytic on

$$A = \mathbb{C} - \left\{ x+iy \;\middle|\; \begin{matrix} y=0 \\ x \le 0 \end{matrix} \right\}$$

$$\frac{d}{dw}\log(w) = \frac{1}{w}$$

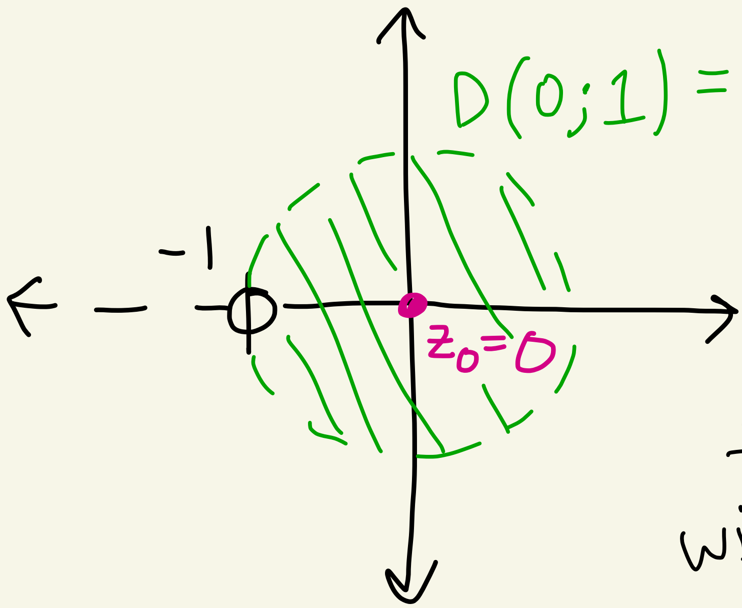


$f(z) = \log(1+z)$ is analytic on

$$B = \mathbb{C} - \left\{ x+iy \;\middle|\; \begin{matrix} x \le -1 \\ y=0 \end{matrix} \right\}$$

What's the Taylor series for f centered at $z_0 = 0$ and on what disc does it converge to $f(z) = \log(1+z)$?

$D(0;1) = \{ z \mid |z| < 1 \}$



By Taylor's theorem the Taylor series for f centered at $z_0 = 0$ will converge to f in $D(0;1)$.

$$f(z) = \log(1+z)$$
$$f'(z) = (1+z)^{-1}$$
$$f''(z) = -(1+z)^{-2}$$
$$f'''(z) = 2(1+z)^{-3}$$
$$f^{(4)}(z) = -3!(1+z)^{-4}$$
$$f^{(5)}(z) = 4!(1+z)^{-5}$$
$$\vdots \qquad \vdots$$
$$\boxed{f^{(k)}(z) = \frac{(-1)^{k-1}(k-1)!}{(1+z)^k} \qquad k \geq 1}$$

Taylor series at $z_0 = 0$ is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^k$$

$$= \sum_{k=1}^{\infty} \frac{\left(\frac{(-1)^{k-1}(k-1)!}{(1+0)^k} \right)}{k!} z^k$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(k-1)!}{k \cdot [(k-1)!]} z^k$$

$$\boxed{\begin{aligned} f^{(0)}(0) &= \log(1+0) \\ &= \log(1) \\ &= 0 \end{aligned}}$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} z^k$$

Thus, $\log(1+z) = \displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} z^k$

for all $z \in D(0;1) = \{ z \mid |z| < 1 \}$

center radius

<u>Ex:</u> For all $z \in \mathbb{C}$ we have

$$\sin(z) = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!}$$

$$= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \frac{z^9}{9!} - \cdots$$

$$\cos(z) = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!}$$

$$= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \frac{z^8}{8!} - \cdots$$

Theorem: Let $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ and $\sum_{n=0}^{\infty} b_n (z-z_0)^n$ be power series with the same center z_0 and radii of convergence $R_1 > 0$ and $R_2 > 0$, respectively. Let $R = \min\{R_1, R_2\}$. Let

$$c_n = \sum_{k=0}^{n} a_k b_{n-k} = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0$$

Then, $\sum_{n=0}^{\infty} c_n (z-z_0)^n$ has radius of convergence $\geq R$ and inside this circle of convergence we have

$$\left(\sum_{n=0}^{\infty} a_n (z-z_0)^n\right)\left(\sum_{n=0}^{\infty} b_n (z-z_0)^n\right) = \sum_{n=0}^{\infty} c_n (z-z_0)^n$$

Hoffman / Marsden book pg 215

$$\left[a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots\right]\left[b_0 + b_1(z-z_0) + b_2(z-z_0)^2 + \dots\right]$$

$$= \underbrace{a_0 b_0}_{c_0} + \underbrace{(a_0 b_1 + a_1 b_0)}_{c_1}(z-z_0)^1$$

$$+ \underbrace{(a_0 b_2 + a_1 b_1 + a_2 b_0)}_{c_2}(z-z_0)^2 + \dots$$

Example to illustrate thm next time...

$f(z) = 1 - \cos(z^5), \quad z_0 = 0$

Then, $f(0) = 1 - \cos(0^5) = 1 - \cos(0) = 1 - 1 = 0$

For all $z \in \mathbb{C}$ we have that

$$f(z) = 1 - \cos(z^5)$$

$$= 1 - \left[1 - \frac{(z^5)^2}{2!} + \frac{(z^5)^4}{4!} - \frac{(z^5)^6}{6!} + \frac{(z^5)^8}{8!} - \dots\right]$$

$$= \frac{z^{10}}{2!} - \frac{z^{20}}{4!} + \frac{z^{30}}{6!} - \frac{z^{40}}{8!} + \dots$$

$$= z^{10}\left[\frac{1}{2!} - \frac{z^{10}}{4!} + \frac{z^{20}}{6!} - \frac{z^{30}}{8!} + \dots\right]$$

$$\underbrace{\qquad\qquad\qquad}_{\varphi(z)}$$

$$\varphi(0) = \frac{1}{2!} \neq 0$$

So,

$$f(z) = z^{10}\varphi(z)$$

where φ is also analytic at 0] more next time

and $\varphi(0) \neq 0$.

Next time we are going to say that f has a zero at $z_0 = 0$ of order 10.