Math 5680
4/10/23

Let's prove two of the theorems from Topic 4

Theorem: Let g and h be analytic at z_0. Suppose g has a zero of order $m \geq 0$ at z_0 and h has a zero of order $k > 0$ at z_0. [If $m=0$, this means $g(z_0) \neq 0$].

(i) If $m \geq k$, then $f(z) = \frac{g(z)}{h(z)}$ has a removable singularity at z_0.

(ii) If $m < k$, then $f(z) = \frac{g(z)}{h(z)}$ has a pole of order $k-m$ at z_0.

proof: We know $g(z) = (z-z_0)^m \varphi_1(z)$ and $h(z) = (z-z_0)^k \varphi_2(z)$ where $\varphi_1(z_0) \neq 0$ and $\varphi_2(z_0) \neq 0$ and φ_1 and φ_2 are analytic at z_0.

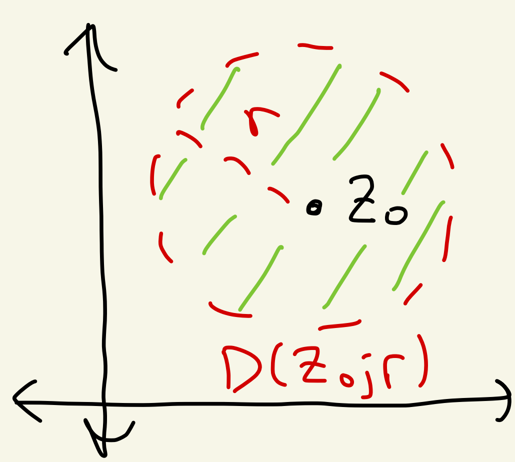
Since φ_1, φ_2 are analytic at z_0 there exists $\hat{r} > 0$ where

φ_1 and φ_2 are analytic on $D(z_0; \hat{r})$.

By Math 4680 since φ_2 is continuous at z_0 and $\varphi_2(z_0) \neq 0$ there exists $\hat{\hat{r}} > 0$ where $\varphi_2(z) \neq 0$ for all $z \in D(z_0; \hat{\hat{r}})$.

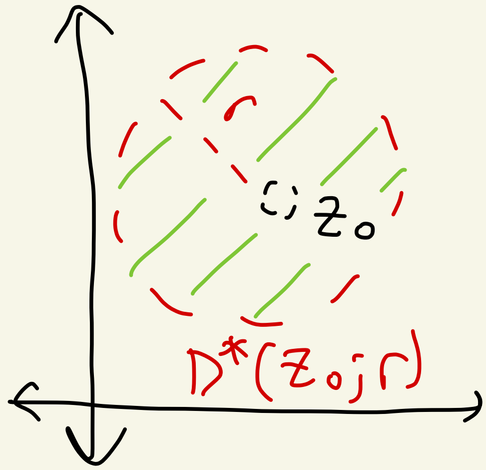
Let $r = \min\{\hat{r}, \hat{\hat{r}}\}$.

So, φ_1, φ_2 are analytic on $D(z_0; r)$ and $\varphi_2(z) \neq 0$ on $D(z_0; r)$.



Thus, if $z \in D^*(z_0; r)$ then

$$f(z) = \frac{g(z)}{h(z)} = \frac{(z-z_0)^m \varphi_1(z)}{(z-z_0)^k \varphi_2(z)}.$$



<u>case (i) – Suppose $m \geq k$.</u>

Then if $z \in D^*(z_0; r)$ then

$$f(z) = (z-z_0)^{m-k}\left[\frac{\varphi_1(z)}{\varphi_2(z)}\right].$$

But also since $m-k \geq 0$ we know

$$(z-z_0)^{m-k}\left[\frac{\varphi_1(z)}{\varphi_2(z)}\right] \text{ is analytic}$$

in <u>all</u> of $D(z_0; r)$.

So, $(z-z_0)^{m-k}\left[\dfrac{\varphi_1(z)}{\varphi_2(z)}\right] = \displaystyle\sum_{n=0}^{\infty} a_n (z-z_0)^n$

for $z \in D(z_0; r)$

But that means

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

for all $z \in D^*(z_0; r)$.

So, this is f's Laurent series at z_0.

So, we have a removable singularity at z_0.

<u>Case (ii) - Suppose $k > m$</u>

Then if $z \in D^*(z_0; r)$ we have

$$f(z) = \frac{(z - z_0)^m \varphi_1(z)}{(z - z_0)^k \varphi_2(z)}$$

$$= \frac{\left(\varphi_1(z) / \varphi_2(z)\right)}{(z - z_0)^{k-m}}$$

Since φ_1, φ_2 are analytic in $D(z_0;r)$ and $\varphi_2(z) \neq 0$ when $z \in D(z_0;r)$ we know $\frac{\varphi_1}{\varphi_2}$ is analytic in $D(z_0;r)$.

So,
$$\frac{\varphi_1(z)}{\varphi_2(z)} = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

for all $z \in D(z_0;r)$.

Also, $a_0 = \frac{\varphi_1(z_0)}{\varphi_2(z_0)} \neq 0$ because $\varphi_1(z_0) \neq 0$.

Thus, if $z \in D^*(z_0;r)$ then

$$f(z) = \frac{\varphi_1(z)/\varphi_2(z)}{(z-z_0)^{k-m}}$$

$$= \frac{1}{(z-z_0)^{k-m}} \left[\sum_{n=0}^{\infty} a_n (z-z_0)^n \right]$$

$$= \frac{1}{(z-z_0)^{k-m}} \left[a_0 + a_1 (z-z_0) + \cdots \right]$$

$$= \frac{a_0}{(z-z_0)^{k-m}} + \frac{a_1}{(z-z_0)^{k-m-1}} + \cdots + \frac{a_{k-m-1}}{(z-z_0)}$$

$$+ a_{k-m} + a_{k-m+1}(z-z_0) + \cdots$$

Since $a_0 \neq 0$ we see we have a pole of order $k-m$.

$\blacksquare$

Theorem:

Let f have an isolated singularity at z_0.

Then z_0 is a pole of order m iff $f(z) = \dfrac{\varphi(z)}{(z-z_0)^m}$

on some $D^*(z_0;r)$ where φ is analytic on $D(z_0;r)$ and $\varphi(z_0) \neq 0$.

Moreover, if this is the case then $\operatorname{Res}(f;z_0) = \dfrac{\varphi^{(m-1)}(z_0)}{(m-1)!}$

proof:

($\Leftarrow$) Suppose there exists r, φ where

$$f(z) = \frac{\varphi(z)}{(z-z_0)^m} \quad \text{for all } z \in D^*(z_0;r)$$

and φ is analytic in $D(z_0;r)$

and $\varphi(z_0) \neq 0$.

From the above we can write

$$\varphi(z) = \sum_{n=0}^{\infty} \frac{\varphi^{(n)}(z_0)}{n!}(z-z_0)^n$$

for all $z \in D(z_0;r)$

So if $z \in D^*(z_0;r)$ then

$$f(z) = \frac{1}{(z-z_0)^m}\varphi(z)$$

$$= \frac{1}{(z-z_0)^m}\left[\sum_{n=0}^{\infty} \frac{\varphi^{(n)}(z_0)}{n!}(z-z_0)^n\right]$$

$$= \frac{1}{(z-z_0)^m}\left[\varphi(z_0) + \frac{\varphi^{(1)}(z_0)}{1!}(z-z_0) + \dots\right]$$

$$= \frac{\varphi(z_0)}{(z-z_0)^m} + \frac{\varphi^{(1)}(z_0)/1!}{(z-z_0)^{m-1}} + \frac{\varphi^{(2)}(z_0)/2!}{(z-z_0)^{m-2}}$$

$$+ \dots + \frac{\varphi^{(m-1)}(z_0)/(m-1)!}{(z-z_0)} + \frac{\varphi^{(m)}(z_0)}{m!}$$

$$+ \frac{\varphi^{(m+1)}(z_0)}{(m+1)!}(z-z_0) + \dots$$

not 0

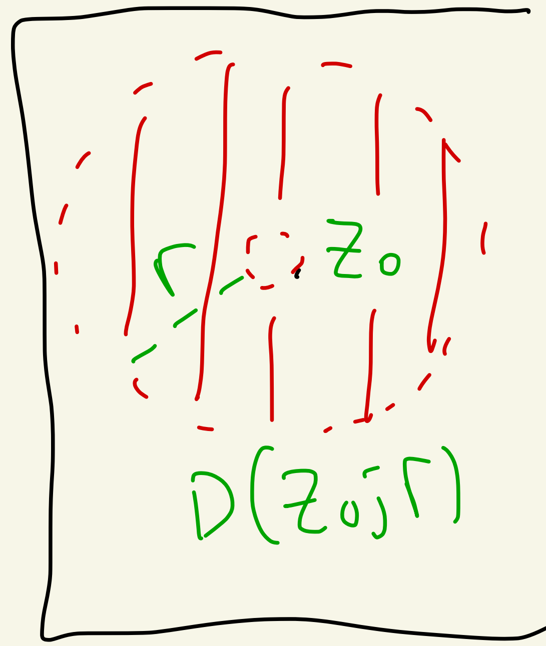
b_1 residue

So, f has a pole of order m at z_0 and $\operatorname{Res}(f; z_0) = \dfrac{\varphi^{(m-1)}(z_0)}{(m-1)!}$

($\Rightarrow$) Suppose f has a pole of order m at z_0.

Thus, there exists an $r>0$ where

$$f(z) = \left[\frac{b_m}{(z-z_0)^m} + \cdots + \frac{b_1}{(z-z_0)}\right] + \sum_{n=0}^{\infty} a_n (z-z_0)^n$$



for all $z \in D^*(z_0; r)$ and where $b_m \neq 0$.

Then if $z \in D^*(z_0; r)$ then

$$(z-z_0)^m f(z) = b_m + b_{m-1}(z-z_0) + \cdots + b_1 (z-z_0)^{m-1} + \sum_{n=0}^{\infty} a_n (z-z_0)^{n+m}$$

For $z \in D(z_0; r)$ set

$$\varphi(z) = b_m + b_{m-1}(z-z_0) + \dots + b_1(z-z_0)^{m-1} + \sum_{n=0}^{\infty} a_n (z-z_0)^{n+m}$$

So φ is analytic in $D(z_0; r)$

and $\varphi(z_0) = b_m \neq 0$.

And,

$$f(z) = \frac{\varphi(z)}{(z-z_0)^m}$$

for $z \in D^*(z_0; r)$. $\blacksquare$
