Math 5680

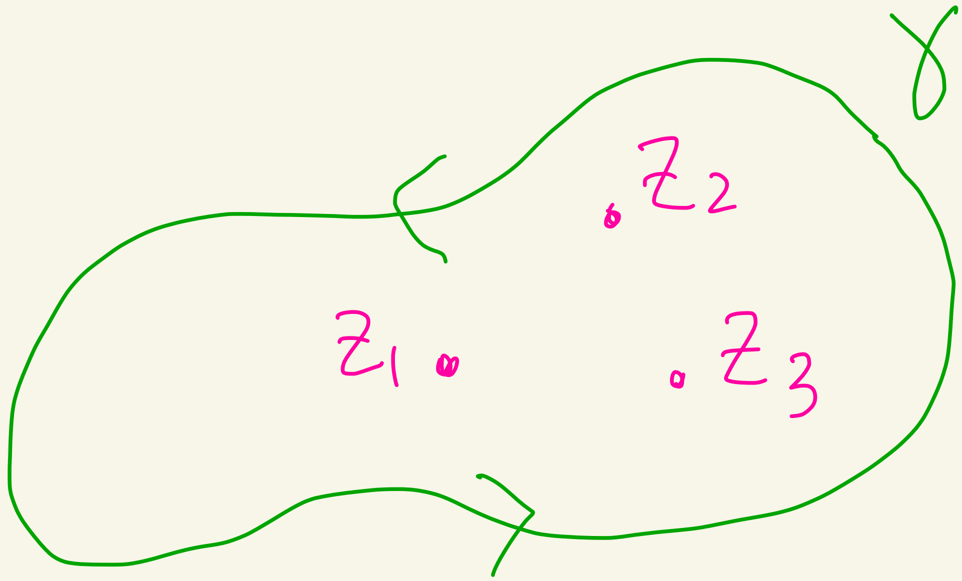
4/12/23

Topic 5 – The Residue Theorem

<u>Theorem</u> (Cauchy's Residue Theorem)
Let γ be a simple, closed, piecewise smooth curve, oriented counterclockwise.
If f is analytic inside and on γ except for a finite number of isolated singularities $z_1, z_2, \ldots, z_n$ of the function f that lie inside γ, then

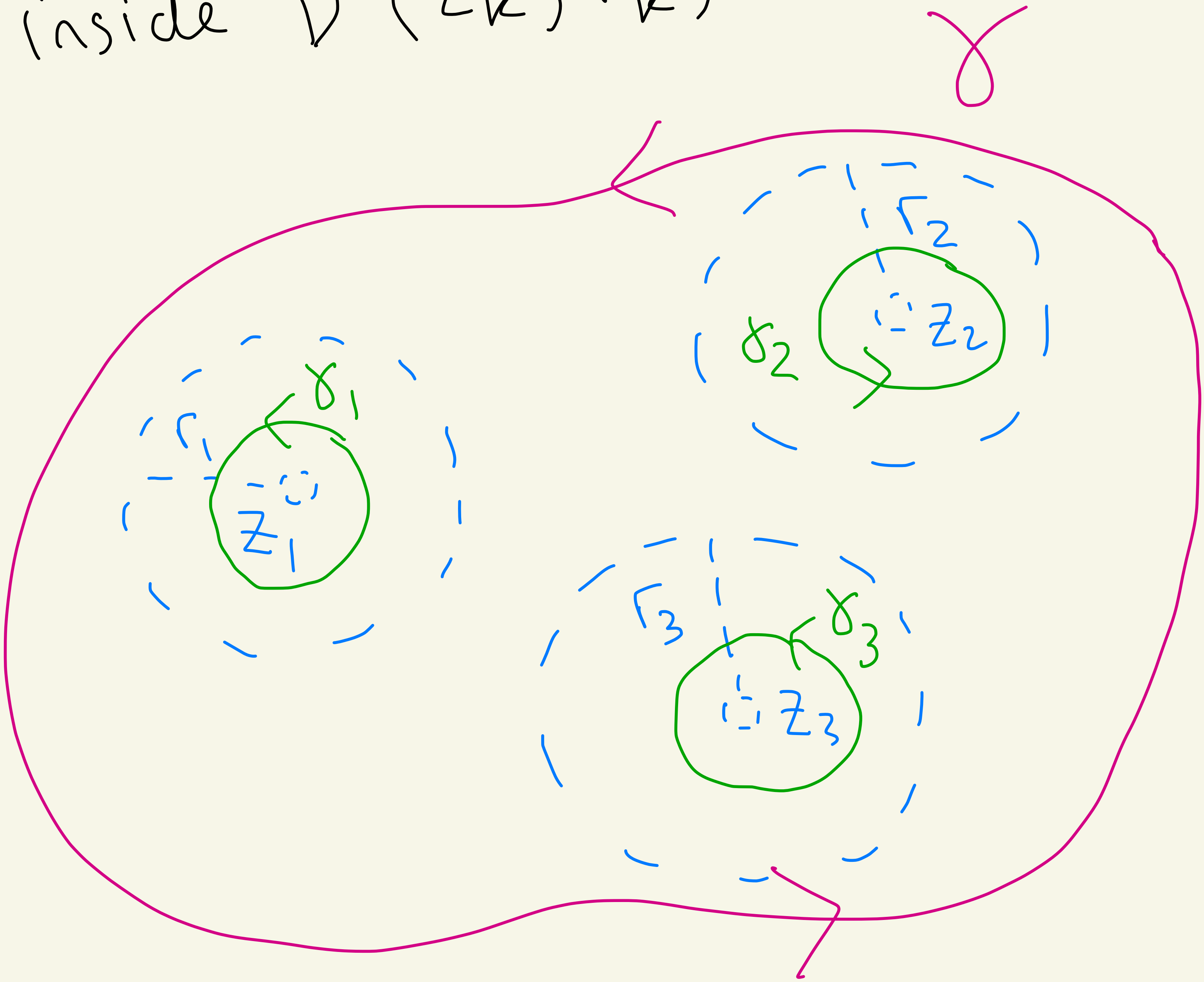
$$\int_\gamma f = 2\pi i \sum_{k=1}^{n} \operatorname{Res}(f; z_k)$$

n=3 picture



<u>proof:</u> For each z_k there is a number $r_k > 0$ where f is analytic on $D^*(z_k; r_k)$ and $D(z_k; r_k)$ lies inside γ. Pick each r_k so that none of these deleted neighborhoods

overlap. Let γ_k be a counter-clockwise oriented circle centered at z_k and contained inside $D^*(z_k; r_k)$.



From Math 4680, since f is analytic on and in-between

γ and $\gamma_1, \gamma_2, \ldots, \gamma_n$ we know

$$\int_\gamma f = \sum_{k=1}^{n} \int_{\gamma_k} f$$

Let's take a look at $\int_{\gamma_k} f$.

Inside of $D^*(z_k, r_k)$ we have get a Laurent series:

$$f(z) = \sum_{n=1}^{\infty} \frac{b_n}{(z-z_k)^n} + \sum_{n=0}^{\infty} a_n (z-z_k)^n$$

$$\boxed{\underline{\text{Recall:}} \quad b_\ell = \frac{1}{2\pi i} \int_{\gamma_k} f(z) \cdot (z-z_k)^{\ell-1} \, dz}$$

$$\ell = 1 \Rightarrow b_1 = \frac{1}{2\pi i} \int_{\gamma_k} f(z) \, dz$$

$$\boxed{b_1 = \mathrm{Res}(f; z_k)}$$

So, $\displaystyle\int_{\gamma_k} f = 2\pi i \,\mathrm{Res}(f; z_k)$

Thus, $$\int f = \sum_{k=1}^{n} \int_{\gamma_k} f$$

$$= 2\pi i \sum_{k=1}^{n} \mathrm{Res}(f; z_k)$$

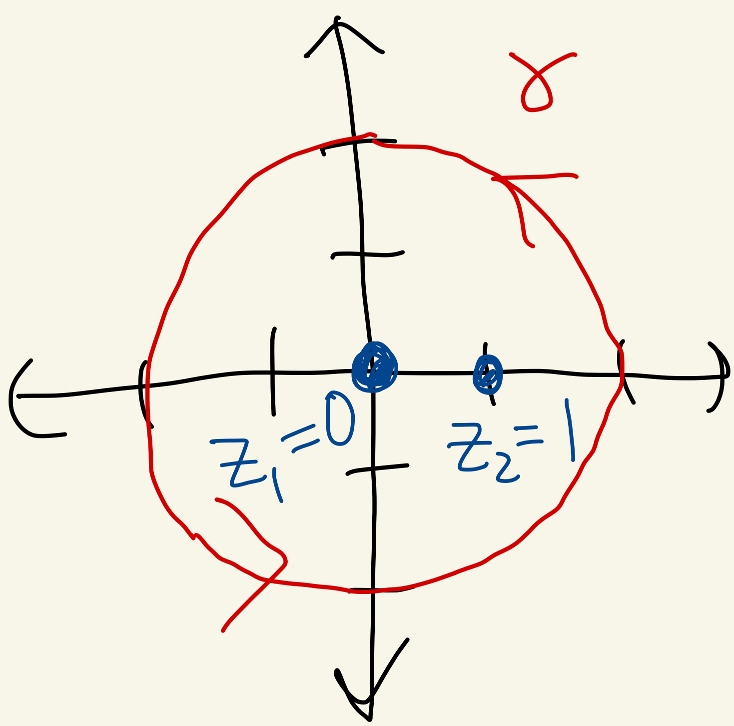
$\blacksquare$

Ex:

Consider $\int_\gamma \frac{5z-2}{z(z-1)}\,dz$

where γ is the circle $|z|=2$ oriented counterclockwise.

Let $f(z)=\frac{5z-2}{z(z-1)}$



f has isolated singularities $z_1=0$, $z_2=1$ inside of γ.

Residue theorem says

$$\int_\gamma f = 2\pi i \operatorname{Res}(f;0) + 2\pi i \operatorname{Res}(f;1)$$

Let calculate $\operatorname{Res}(f;0)$ first

Note that

$$f(z) = \frac{5z-2}{z(z-1)} = \frac{\left(\frac{5z-2}{z-1}\right)}{z} = \frac{\varphi(z)}{z}$$

where $\varphi(z) = \frac{5z-2}{z-1}$ is analytic at 0 and $\varphi(0) = \frac{-2}{-1} = 2 \neq 0$.

From our theorem in class we have a pole of order $m=1$

And $\operatorname{Res}(f;0) = \dfrac{\varphi^{(m-1)}(0)}{(m-1)!}$

$$= \frac{\varphi(0)}{0!}$$

$$= \varphi(0) = 2$$

<u>Let's calculate $\text{Res}(f;1)$</u>

We have

$$f(z) = \frac{5z-2}{z(z-1)} = \frac{5z-2}{z^2-z} = \frac{g(z)}{h(z)}$$

where $g(z) = 5z-2$, $h(z) = z^2 - z$

g and h are analytic at 1.

$g(1) = 3 \neq 0$

$\boxed{h'(z) = 2z-1}$

$h(1) = 0$

$h'(1) = 2(1) - 1 = 1 \neq 0$

So, f has a simple pole at 1.

And, $\operatorname{Res}(f;1) = \dfrac{g(1)}{h'(1)} = \dfrac{3}{1} = 3$

Thus,

$$\int_\gamma f = 2\pi i \left[\operatorname{Res}(f;0) + \operatorname{Res}(f;1) \right]$$

$$= 2\pi i \left[2+3 \right] = \boxed{10\pi i}$$

Topic 6 - Applications to integrals

from notes

Application II - Improper integrals

Recall if $f(x)$ is a real-valued function for $x \in \mathbb{R}$ that is defined for $x \geq a$ then

$$\boxed{\int_a^\infty f(x)\,dx = \lim_{R \to \infty} \int_a^R f(x)\,dx}$$

Similarly if f is defined for $x \leq a$ then

$$\boxed{\int_{-\infty}^{a} f(x)\,dx = \lim_{R\to\infty} \int_{-R}^{a} f(x)\,dx}$$

If f is defined for all $x \in \mathbb{R}$

then

$$\int_{-\infty}^{\infty} f(x)\,dx = \left[\lim_{R\to\infty} \int_{-R}^{a} f(x)\,dx\right] + \left[\lim_{R\to\infty} \int_{a}^{R} f(x)\,dx\right]$$

for any $a \in \mathbb{R}$.

$\int_{-\infty}^{\infty} f(x)\,dx$ exists iff both integrals on the right exist.

Fact: Suppose $f:\mathbb{R}\to\mathbb{R}$ is even [that is, $f(-x)=f(x)$ for all $x\in\mathbb{R}$]

If the Cauchy principal value of f [which is $\lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx$]

exists, then $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^a f(x)\,dx$

exist for any $a\in\mathbb{R}$ and

$$2\int_0^\infty f(x)\,dx = \int_{-\infty}^\infty f(x)\,dx = \lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx$$

proof: Since f is even,

$$\int_{-R}^{0} f(x)\,dx = \int_0^R f(x)\,dx = \frac{1}{2}\int_{-R}^{R} f(x)\,dx$$

Take $R\to\infty$. $\blacksquare$

Ex: Let's calculate

$$\int_0^\infty \frac{x^2}{x^6+1}\,dx$$

Let $f(x) = \frac{x^2}{x^6+1}$.

Then, $f(-x) = \frac{(-x)^2}{(-x)^6+1} = f(x)$.

So, f is even.

Thus,

$$\int_0^\infty \frac{x^2}{x^6+1}\,dx = \frac{1}{2} \lim_{R\to\infty} \left[\int_{-R}^{R} \frac{x^2}{x^6+1}\,dx \right]$$

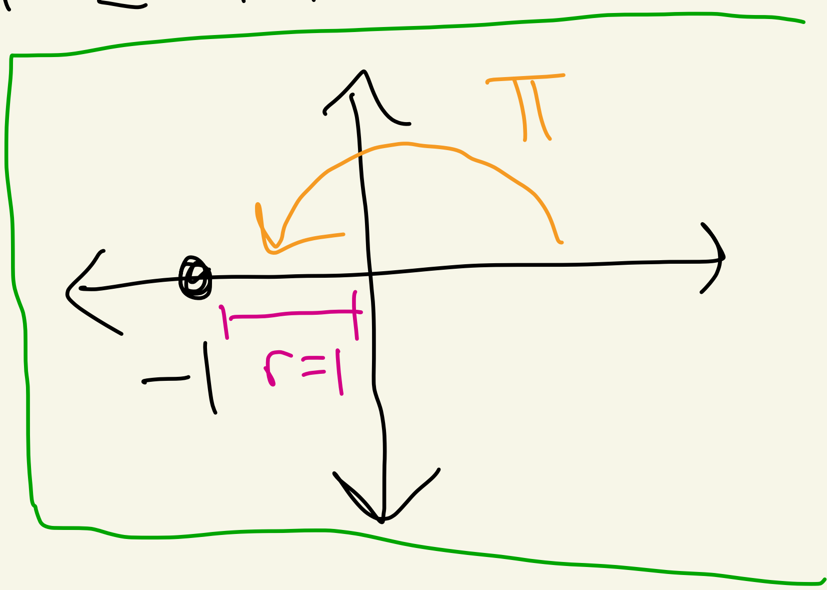
Think of f as being a complex function now, ie

$$f(z) = \frac{z^2}{z^6+1}.$$

Let's find the singularities of f.

These occur when $z^6+1=0$.

$$z^6 = -1 = 1 \cdot e^{\pi i}$$



roots:

$$z_k = 1^{1/6} \cdot e^{\left(\frac{\pi}{6}+\frac{2\pi k}{6}\right)i}, \quad k=0,1,2,3,4,5$$

$z_0 = e^{i\pi/6}$

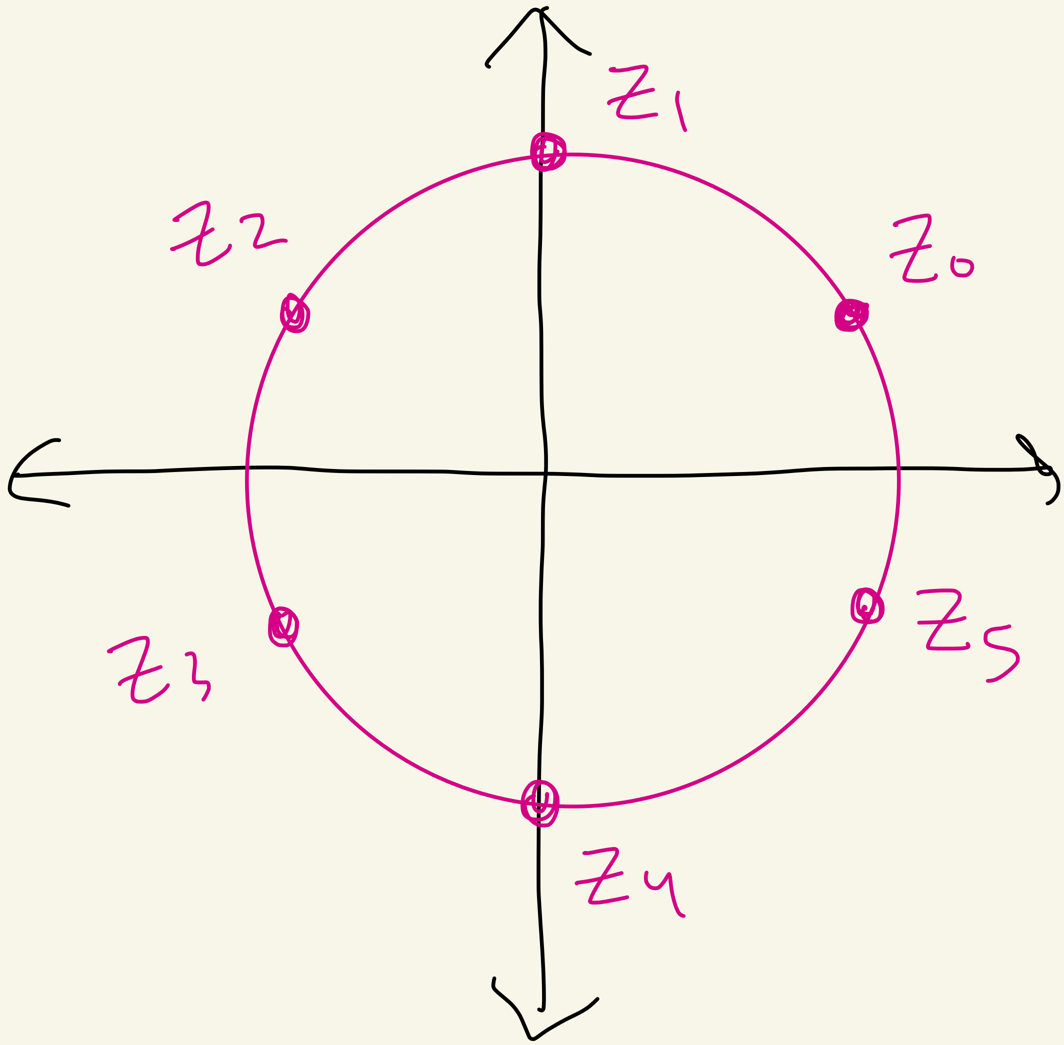
$z_1 = e^{i3\pi/6}$

$z_2 = e^{i5\pi/6}$

$z_3 = e^{i7\pi/6}$

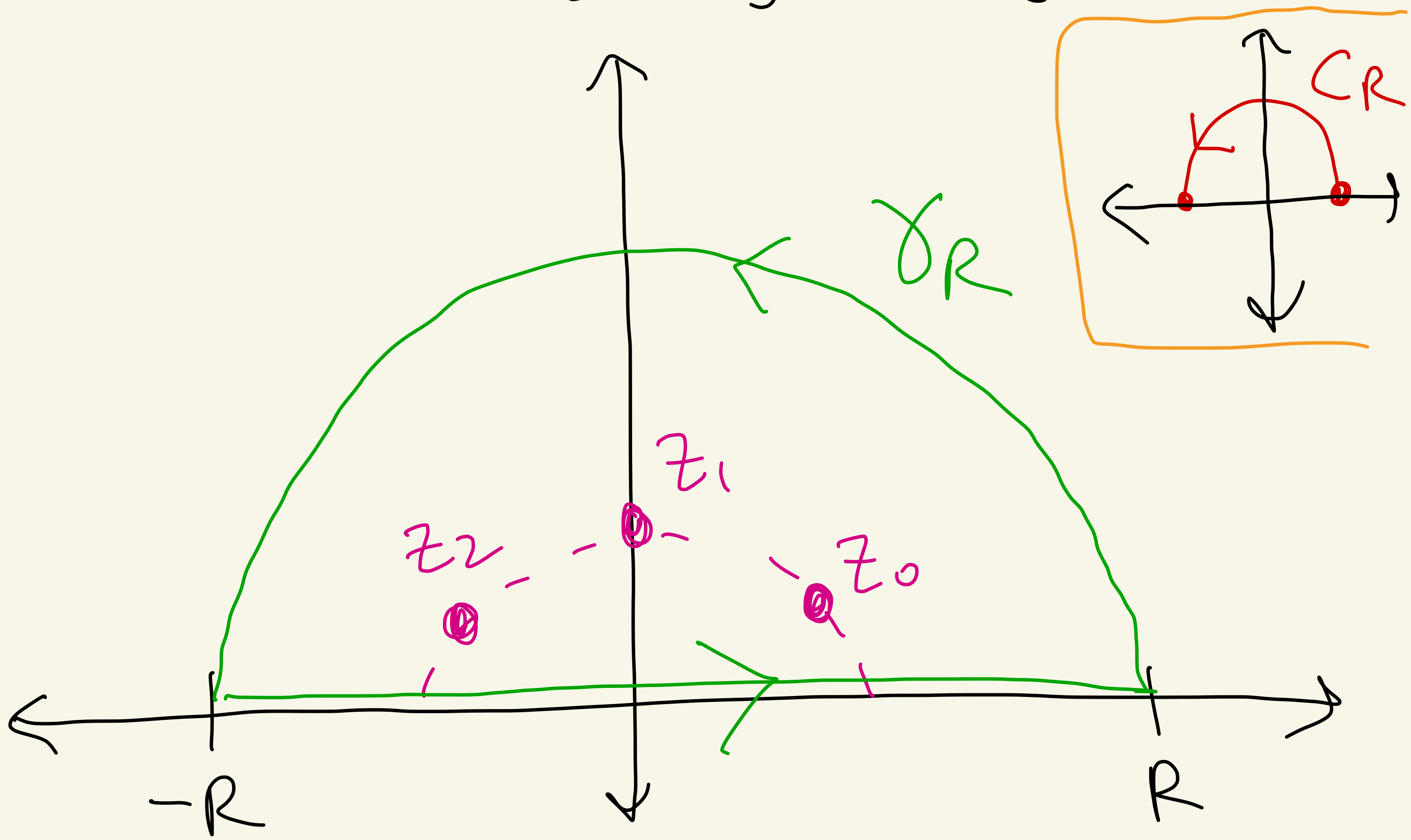
$z_4 = e^{i9\pi/6}$

$z_5 = e^{i11\pi/6}$



Given $R > 1$, let C_R be the upper half of the circle $|z| = R$ oriented counterclockwise. Let γ_R be the closed curve formed by going along the

x-axis from $-R$ to R
and then going along C_R.



Thus,

$$\int_{\gamma_R} f(z)\,dz = \int_{C_R} f(z)\,dz + \underbrace{\int_{-R}^{R} f(x)\,dx}_{\text{this part is a real integral } z=x \text{ where } -R\le x\le R}$$

We will calculate

$$\int_{\gamma_R} f(z)\,dz = 2\pi i \sum_{k=1}^{3} \operatorname{Res}(f; z_k)$$

and we will show

$$\lim_{R\to\infty} \int_{C_R} f(z)\,dz = 0.$$

This will allow us to calculate

$$\lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx = 2\pi i \sum_{k=1}^{3} \operatorname{Res}(f; z_k)$$