Math 5680
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Improper integrals involving sine and cosine

Sometimes if we want to evaluate

$$\int_{-\infty}^{\infty} f(x)\sin(ax)\,dx \quad \text{or} \quad \int_{-\infty}^{\infty} f(x)\cos(ax)\,dx$$

where $x \in \mathbb{R}$, $f(x) \in \mathbb{R}$, $a > 0$, $a \in \mathbb{R}$,

then we can use

$$\int_{-R}^{R} f(x)\cos(ax)\,dx + i\int_{-R}^{R} f(x)\sin(ax)\,dx$$

$$= \int_{-R}^{R} f(x)e^{iax}\,dx$$

$$\boxed{e^{iax} = \cos(ax) + i\sin(ax)}$$

together with the fact that

$|e^{iaz}| = e^{-ay}$ is bounded when $y \geq 0$

Ex: Let's show that

$$\boxed{\int_{-\infty}^{\infty} \frac{\cos(3x)}{(x^2+1)^2}\,dx = \frac{2\pi}{e^3}}$$

$\cos(-\theta) = \cos(\theta)$

Note that

$$\frac{\cos(3(-x))}{((-x)^2+1)^2} = \frac{\cos(3x)}{(x^2+1)^2} \quad \text{for all } x$$

So we have an even function. Thus,

$$\int_{-\infty}^{\infty} \frac{\cos(3x)}{(x^2+1)^2}\,dx = \underbrace{\lim_{R\to\infty} \int_{-R}^{R} \frac{\cos(3x)}{(x^2+1)^2}\,dx}_{\text{Cauchy principle value}}$$

Consider the function $\dfrac{e^{i3z}}{(z^2+1)^2}$

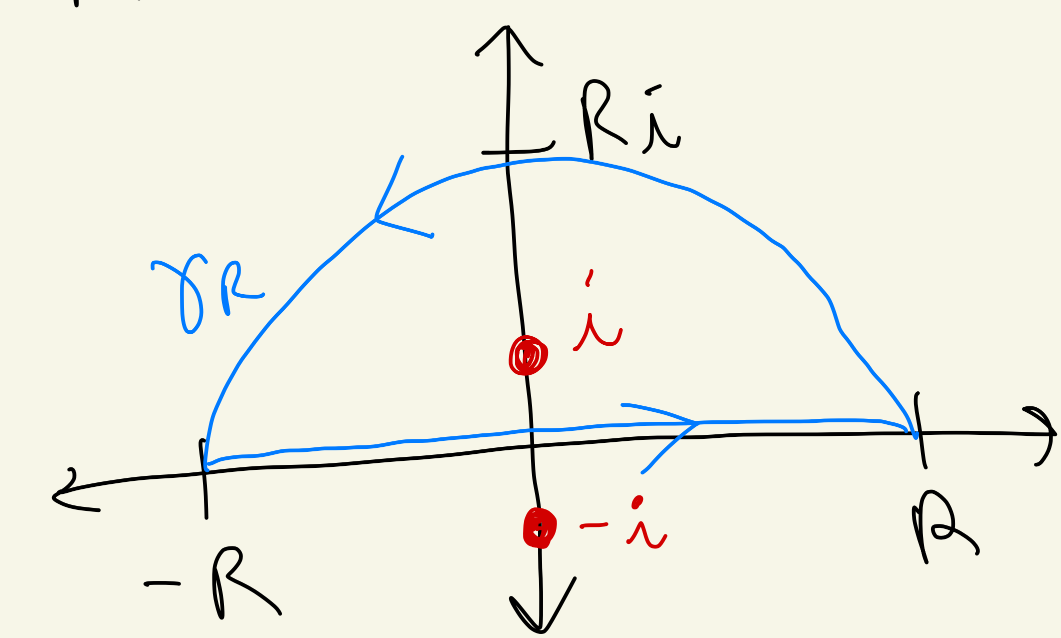
On the real axis when $z = x$ we get

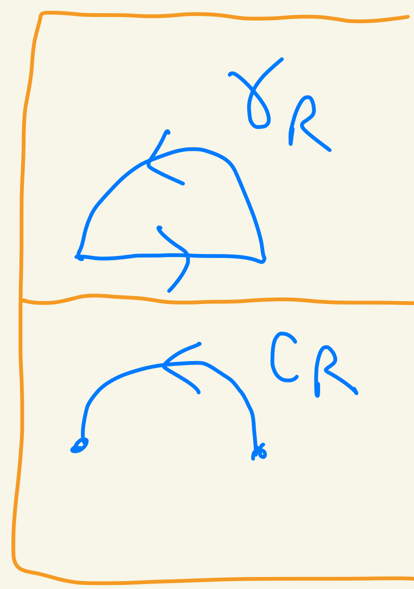
$$\mathrm{Re}\left(\frac{e^{i3z}}{(z^2+1)^2}\right) = \mathrm{Re}\left(\frac{e^{i3x}}{(x^2+1)^2}\right)$$

$$= \mathrm{Re}\left(\frac{\cos(3x)+i\sin(3x)}{(x^2+1)^2}\right) = \frac{\cos(3x)}{(x^2+1)^2}$$

Note that $\dfrac{e^{i3z}}{(z^2+1)^2}$ is analytic everywhere except when $z^2+1=0$ which is when $z=\pm i$.

Let $R>1$ and C_R be the top half of the circle of radius R oriented counter-clockwise. Let γ_R be the straight line from $-R$ to R followed by C_R.





By the residue theorem,

$$\int_{\gamma_R} \frac{e^{i3z}}{(z^2+1)^2}\,dz = 2\pi i \operatorname{Res}\left(\frac{e^{i3z}}{(z^2+1)^2}; i\right)$$

$\varphi(z)$

And we have

$$\frac{e^{i3z}}{(z^2+1)^2} = \frac{e^{i3z}}{\left((z+i)(z-i)\right)^2} = \frac{\left(\frac{e^{i3z}}{(z+i)^2}\right)}{(z-i)^2}$$

$$= \frac{\varphi(z)}{(z-i)^2}$$

φ is analytic at i

and $\varphi(i) = \dfrac{e^{i3(i)}}{(2i)^2} = \dfrac{e^{-3}}{-4} \neq 0$

Thus, we have a pole of order 2 at i and

$$\operatorname{Res}\left(\frac{e^{i3z}}{(z^2+1)^2}; i\right) = \frac{\varphi^{(2-1)}(i)}{(2-1)!} = \varphi'(i)$$

We have $\varphi(z) = (z+i)^{-2} \cdot e^{i3z}$

and $\varphi'(z) = -2(z+i)^{-3} e^{i3z} + (z+i)^{-2} e^{i3z} (3i)$

So, $\varphi'(i) = -2(2i)^{-3} e^{-3} + (2i)^{-2}(3i) e^{-3}$

$$= e^{-3}\left[\frac{-2}{(2i)^3} + \frac{3i}{(2i)^2}\right]$$

$$= \frac{1}{e^3}\left[\frac{-2}{-8i} + \frac{3i}{-4}\right]$$

$\frac{1}{i} = -i$

$$= \frac{1}{e^3}\left[-\frac{1}{4}i - \frac{3}{4}i\right] = \frac{-i}{e^3}$$

Thus,

$$\boxed{\int_{\gamma_R} \frac{e^{i3z}}{(z^2+1)^2}\,dz} = 2\pi i\left(\frac{-i}{e^3}\right) = \frac{2\pi}{e^3}$$

So,

$$\boxed{\underbrace{\int_{-R}^{R} \frac{e^{i3x}}{(x^2+1)^2}\,dx}_{\text{on the real axis, } z=x} + \int_{C_R} \frac{e^{i3z}}{(z^2+1)^2}\,dz} = \frac{2\pi}{e^3} \qquad (*)$$

equal

Note

$$\int_{-R}^{R} \frac{e^{i3x}}{(x^2+1)^2}\,dx = \int_{-R}^{R} \frac{\cos(3x)}{(x^2+1)^2}\,dx + i\int_{-R}^{R} \frac{\sin(3x)}{(x^2+1)^2}\,dx$$

$$\boxed{e^{i3x} = \cos(3x) + i\sin(3x)}$$

Taking the real part of $(*)$ we get

$$\int_{-R}^{R} \frac{\cos(3x)}{(x^2+1)^2}\,dx + \operatorname{Re}\left[\int_{C_R} \frac{e^{i3z}}{(z^2+1)^2}\,dz\right] = \frac{2\pi}{e^3} \qquad (**)$$

Goal: Show $\lim_{R\to\infty} \int_{C_R} \frac{e^{i3z}}{(z^2+1)^2}\,dz = 0.$

Let $z = x+iy$ live on C_R.

So, $y \geq 0$.

Then, $|z| = R$.

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$|z+w| \geq ||z|-|w||$

Then,

$$|z^2+1| \geq ||z^2|-|1||$$
$$= ||z|^2-1|$$
$$= |R^2-1|$$
$$= R^2-1$$

$R>1$
$R^2>1$
$R^2-1>0$

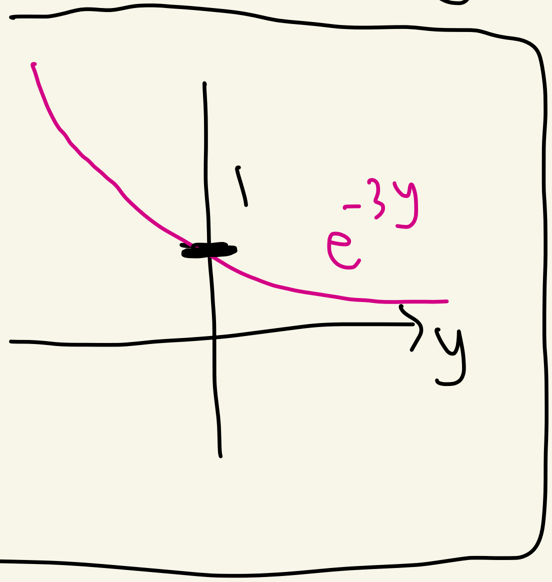
So, $\frac{1}{|z^2+1|^2} \leq \frac{1}{(R^2-1)^2}$.

Also,

$$|e^{i3z}| = |e^{i3(x+iy)}| = |e^{3ix-3y}|$$

$$= \underbrace{|e^{i3x}|}_{1}|e^{-3y}| = |e^{-3y}|$$

$$= e^{-3y} = \frac{1}{e^{3y}} \le 1 \quad \text{since } y \geq 0.$$



So, in summary if $z \in C_R$ then

$$\left|\frac{e^{i3z}}{(z^2+1)^2}\right| \le \frac{1}{(R^2-1)^2}$$

So,

$$\left|\int_{C_R} \frac{e^{i3z}}{(z^2+1)^2}\,dz\right| \le \frac{1}{(R^2-1)^2} \cdot \underbrace{\pi R}_{\text{arclength of } C_R}$$

$$= \frac{\pi R}{R^4 - 2R^2 + 1}$$

$$= \frac{\pi / R^3}{1 - 2/R^2 + 1/R^4}$$

$$\longrightarrow \frac{0}{1 - 0 + 0} = 0$$

as $R \to \infty$.

Let $R \to \infty$ in $(**)$ and you get that

$$\int_{-\infty}^{\infty} \frac{\cos(3x)}{(x^2+1)^2}\, dx = \frac{2\pi}{e^3} \quad \blacksquare$$