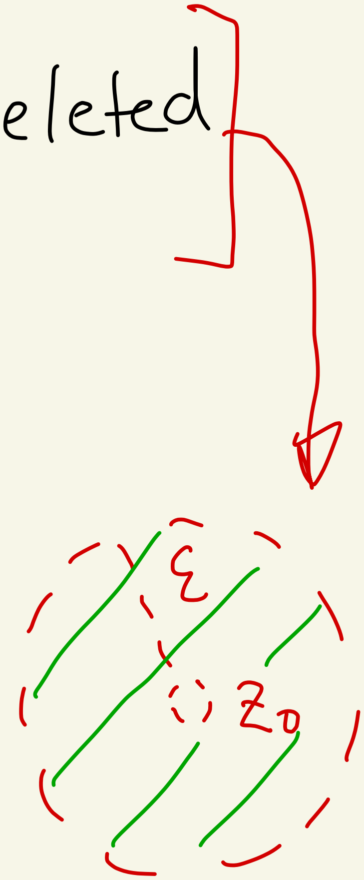
Math 5680
4/3/23

Theorem (Removable Singularity Thm)

Let $z_0 \in \mathbb{C}$. Suppose z_0 is an isolated singularity of f.

Then, z_0 is a removable singularity of f iff one of the following conditions holds:

① f is bounded in some deleted ε-neighborhood of z_0



② $\lim_{z \to z_0} f(z)$ exists

③ $\lim_{z \to z_0} (z-z_0) f(z) = 0$

Ex: $f(z) = \dfrac{\sin(z)}{z}$

$z_0 = 0$ is an isolated singularity.

Using ③ above we see

$$\lim_{z\to 0} (z-0)\cdot \frac{\sin(z)}{z}$$

$$= \lim_{z\to 0} \cancel{z}\cdot \frac{\sin(z)}{\cancel{z}}$$

$\boxed{\text{sin(z) is continuous at 0}}$

$$= \lim_{z\to 0} \sin(z) = \sin(0) = 0$$

So we have a removable singularity at $z_0 = 0$.

Note that if $z \neq 0$, then

$$\frac{\sin(z)}{z} = \frac{1}{z}\left[z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 - \frac{1}{7!}z^7 + \cdots\right]$$

$$= 1 - \frac{1}{3!}z^2 + \frac{1}{5!}z^4 - \frac{1}{7!}z^6 + \cdots$$

Define $\tilde{f}: \mathbb{C} \to \mathbb{C}$ where

$$\tilde{f}(z) = 1 - \frac{1}{3!}z^2 + \frac{1}{5!}z^4 - \frac{1}{7!}z^6 + \dots$$

Note that

$$\tilde{f}(z) = \begin{cases} \frac{\sin(z)}{z} & \text{when } z \neq 0 \\ 1 & \text{when } z = 0 \end{cases}$$

The power series for $\tilde{f}$ converges on all of $\mathbb{C}$.

So, $\tilde{f}$ is analytic on all of $\mathbb{C}$.

It removes the singularity at $z_0 = 0$.

Theorem: Let g and h each be analytic at z_0.

Suppose g has a zero of order $m \geq 0$ at z_0 and h has a zero of order $k > 0$ at z_0.

[If $m = 0$, this means $g(z_0) \neq 0$]

(i) If $m \geq k$, then $f(z) = \frac{g(z)}{h(z)}$ has a removable singularity at z_0.

(ii) If $m < k$, then $f(z) = \frac{g(z)}{h(z)}$ has a pole of order $k - m$ at z_0.

Ex: Let

$$f(z) = \frac{\sin(z)}{z} \begin{array}{l} \leftarrow g(z) = \sin(z) \\ \leftarrow h(z) = z \end{array}$$

Let $z_0 = 0$.

Note $g(0) = \sin(0) = 0$

and $h(0) = 0$

We have an isolated singularity at $z_0 = 0$.

Recall if $z \neq 0$, then

$$\sin(z) = z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 - \frac{1}{7!}z^7 + \cdots$$

$$= z\left(1 - \frac{1}{3!}z^2 + \frac{1}{5!}z^4 - \frac{1}{7!}z^6 + \cdots\right)$$

$$= z\,\varphi(z)$$

where $\varphi(0) \neq 0$.

So, g has a zero of order $m=1$ at $z_0=0$.

And

$$h(z)=z$$

has a zero of order $k=1$ at $z_0=0$.

Thus, $m \geq k$ and the theorem says we have a removable singularity.

Ex: Let $f(z) = \frac{z}{(e^z-1)^2} = \frac{g(z)}{h(z)}$

Let $g(z)=z$
and $h(z)=(e^z-1)^2$

$$\left.\begin{array}{l} g(0)=0 \\ h(0)=0 \end{array}\right.$$

f has an isolated singularity at $z_0=0$.

g has a zero of multiplicity $m=1$ at $z_0=0$.

What about h? We need h's power series centered at $z_0=0$. It is:

$$h(z)=\left(e^{z}-1\right)^{2}$$

$$=\left(-1+e^{z}\right)^{2}$$

$$=\left(\cancel{-1}+\cancel{1}+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\frac{z^4}{4!}+\dots\right)^{2}$$

$$=\left(z+\frac{z^2}{2}+\frac{z^3}{6}+\frac{z^4}{24}+\dots\right)^{2}$$

$$=\left(z+\frac{z^2}{2}+\frac{z^3}{6}+\dots\right)\left(z+\frac{z^2}{2}+\frac{z^3}{6}+\dots\right)$$

$$= z^2 + \left(\frac{1}{2} + \frac{1}{2}\right) z^3 + \left(\frac{1}{6} + \frac{1}{4} + \frac{1}{6}\right) z^4 + \ldots$$

$$= z^2 + z^3 + \frac{7}{12} z^4 + \ldots$$

$$= z^2 \underbrace{\left[1 + z + \frac{7}{12} z^2 + \ldots \right]}_{\varphi(z)}$$

$$= z^2 \varphi(z) \text{ , where } \varphi(0) \neq 0.$$

So, $h(z) = (e^z - 1)^2 = z^2 \varphi(z)$ has a zero of order $k=2$ at $z_0 = 0$.

Summary,

$$f(z) = \frac{z}{(e^z - 1)^2}$$

$\leftarrow$ zero of order $m=1$ at $z_0=0$

$\leftarrow$ zero of order $k=2$ at $z_0=0$.

So, f has a pole of order $k-m = 2-1 = 1$ at $z_0 = 0$.

called a simple pole

But what's the residue?

The next theorem will help.

Theorem (On poles of order m)

Let f have an isolated singularity at $z_0 \in \mathbb{C}$.

Then z_0 is a pole of order m iff $f(z)$ can be written in the form $f(z) = \frac{\varphi(z)}{(z-z_0)^m}$

for all z in some deleted neighborhood $D^*(z_0; r)$, where φ is analytic at z_0 and $\varphi(z_0) \neq 0$.

Moreover if this is the case then

$$\text{Res}(f; z_0) = \frac{\varphi^{(m-1)}(z_0)}{(m-1)!}$$

Ex: $f(z) = \dfrac{z}{(e^z-1)^2}$

Suppose $z \neq 0$.

Then,

previous example

$$\frac{z}{(e^z-1)^2} = \frac{z}{\left(z^2+z^3+\frac{7}{12}z^4+\ldots\right)}$$

$$= \frac{z}{z^2\left(1+z+\frac{7}{12}z^2+\ldots\right)}$$

$$= \frac{1}{z\left(1+z+\frac{7}{12}z^2+\ldots\right)}$$

$$= \frac{\boxed{1/\left(1+z+\frac{7}{12}z^2+\ldots\right)}}{z}$$

$\varphi(z)$

$\varphi(0)=1$

φ is analytic at 0

$$= \frac{\varphi(z)}{z}$$

Theorem says we have a pole of order $m=1$, and

$$\mathrm{Res}(f;0) = \frac{\varphi^{(m-1)}(0)}{(m-1)!}$$

$$= \frac{\varphi^{(0)}(0)}{0!}$$

$$= \varphi(0)$$

$$= \frac{1}{1+0+\frac{7}{12}\cdot 0^2+\cdots}$$

$$= 1$$

$$\begin{array}{r l} & 1 - z + \frac{5}{12}z^2 + \cdots \\ \cline{2-2} 1+z+\frac{7}{12}z^2+\cdots \,\big) & 1 \\ & -\left(1+z+\frac{7}{12}z^2+\cdots\right) \\ \cline{2-2} & -z-\frac{7}{12}z^2-\cdots \\ & -\left(-z-z^2-\cdots\right) \\ \cline{2-2} & \frac{5}{12}z^2+\cdots \end{array}$$

$$\frac{\left(\frac{1}{1+z+\frac{7}{12}z^2+\cdots}\right)}{z} = \frac{1-z+\frac{5}{12}z^2+\cdots}{z}$$

$$= \frac{\boxed{1}}{z} - 1 + \frac{5}{12}z + \cdots$$

Residue