5680
5/10/23

Ex: Let

$$g(z) = \sum_{n=1}^{\infty} \frac{2^n}{z^n}$$

Show g is analytic on

$$A = \{ z \mid 2 < |z| \}.$$

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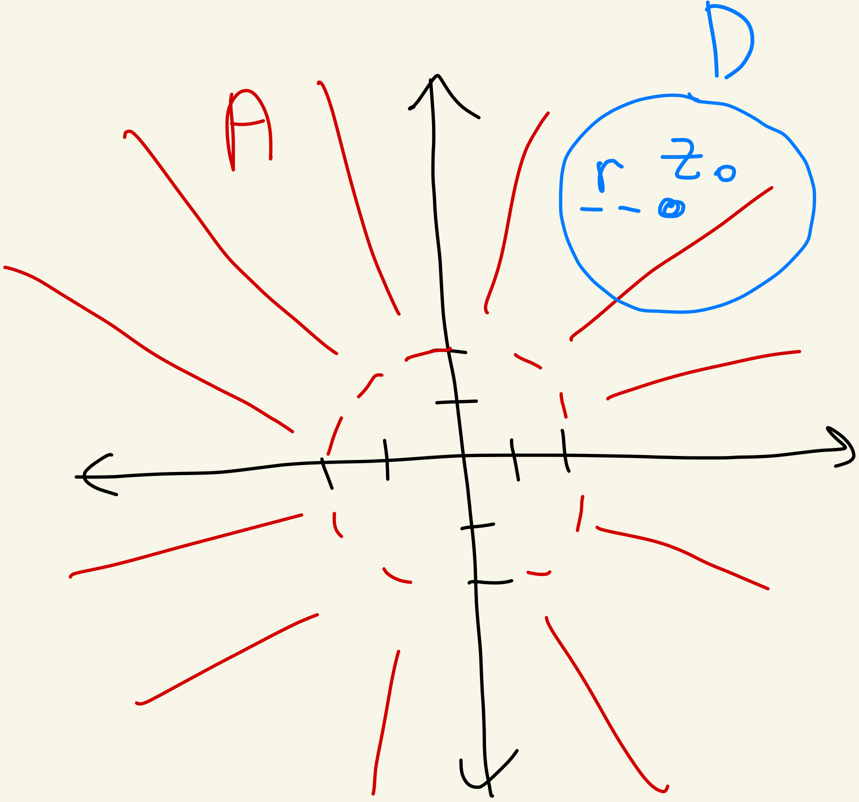
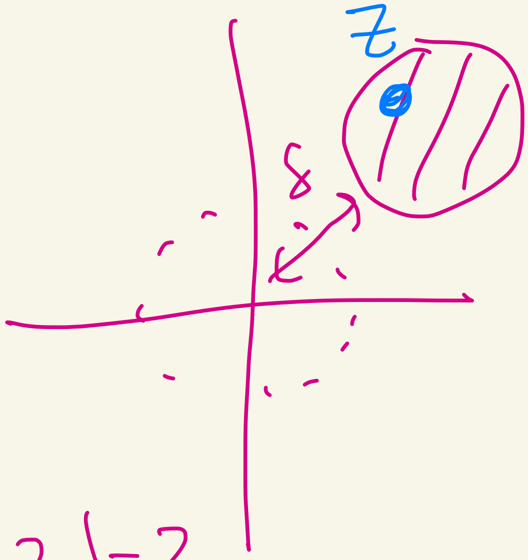
proof:

Let $D = \overline{D(z_0; r)} \subseteq A$ be a closed disc.



Need to show that g(z) converges uniformly on D.



Let $\delta = |z_0| - r = |z_0| - 2$

Then if $z \in D$, then $|z| \geqslant \delta > 2$

So if $z \in D$, then

$$\left|\frac{2^n}{z^n}\right| = \frac{2^n}{|z|^n} \leq \frac{2^n}{\delta^n} = \left(\frac{2}{\delta}\right)^n = M_n$$

Note: Since $0 < \frac{2}{\delta} < 1$ because $2 < \delta$.

So, $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \left(\frac{2}{\delta}\right)^n$ is geometric and converges.

By the WM-test since $\left|\frac{2^n}{z^n}\right| \le M_n$ if $z \in D$ and $\sum M_n$ converges, we know $g(z) = \sum_{n=1}^{\infty} \frac{2^n}{z^n}$ converges uniformly on D.

By Analytic convergence thm, $g(z) = \sum_{n=1}^{\infty} \frac{2^n}{z^n}$ is analytic in A.

▨

If $z \in A$, then

$$g'(z) = \sum_{n=1}^{\infty} \left(2^n z^{-n}\right)' = \sum_{n=1}^{\infty} \frac{-n 2^n}{z^{n+1}}$$

▨

HW 4 - Part 2

① (g) $f(z) = \left(\frac{\cos(z)-1}{z}\right)^2$, $z_0 = 0$

$$h(z) = \sum_{n=0}^{\infty} \frac{h^{(n)}(z_0)}{n!}(z-z_0)^n$$

$h(z_0) = 0$

let $k \geq 1$ be the first time $h^{(k)}(z_0) \neq 0$. Then h has a zero of multiplicity k at z_0

$$f(z) = \frac{(\cos(z)-1)^2}{z^2} = \frac{g(z)}{h(z)}$$

$$g(0) = \cos(0) - 1 = 1 - 1 = 0$$

$$g'(0) = 2(\cos(0)-1)\cdot(-\sin(0)) = 0$$

$$g''(0) =$$

$$\uparrow$$

$$g'(z) = -2\sin(z)(\cos(z)-1)$$

$$g''(z) = -2\cos(z)(\cos(z)-1)$$
$$-2\sin(z)[2(\cos(z)-1)(-\sin(z)))$$

$$g(z) = (\cos(z)-1)^2$$

$$= \left(-1+1-\frac{z^2}{2!}+\frac{z^4}{4!}-\dots\right)^2$$

$$= \left(-\frac{z^2}{2!}+\frac{z^4}{4!}-\dots\right)^2$$

$$= \frac{z^4}{4}+\dots\dots = z^4\left(\frac{1}{4}+\dots\right)$$

g has a zero of order 4 at $z_0 = 0$

$$g''(0) = 0$$

$g'''(0) = 0$

$g''''(0) \neq 0$

g has a zero of order 4 at $z_0 = 0$

$h(z) = z^2$

$h'(0) = 2(0) = 0$

$h''(0) = 2 \neq 0$

h has a zero of order 2 at $z_0 = 0$

$$f(z) = \frac{(\cos(z) - 1)^2}{z^2} \quad \begin{matrix} \leftarrow \text{zero of order 4} \\ \leftarrow \text{zero of order 2} \end{matrix}$$

f has a removable singularity at $z_0 = 0$.

So, $\operatorname{Res}(f;0) = 0$

$$\frac{g}{h} \longleftarrow \frac{g(z_0) \neq 0}{\begin{array}{c} h(z_0) = 0 \\ h'(z_0) \neq 0 \end{array}}$$

$$f(z) = \frac{\sin\left(\frac{\pi}{i} z\right)}{z^4 - 1} = \frac{g(z)}{h(z)}$$

$$\boxed{\begin{aligned} g(z) &= \sin\left(\tfrac{\pi}{i} z\right) \\ h(z) &= z^4 - 1 \end{aligned}}$$

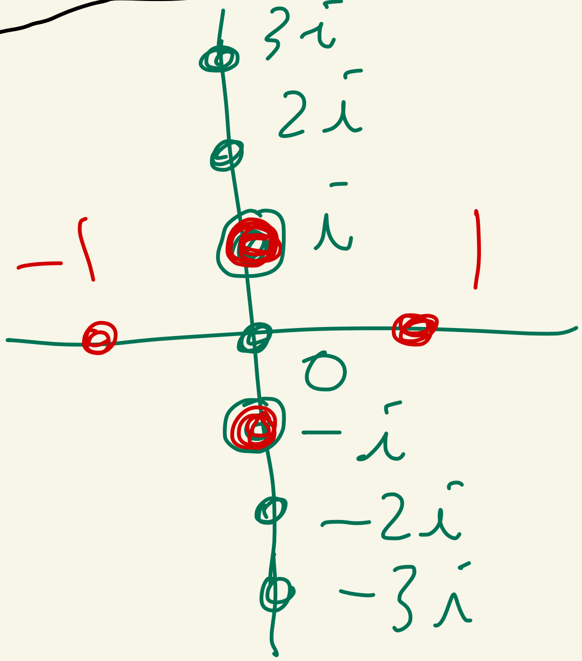
$g(z) = 0$ iff $\frac{\pi}{i} z = \pi k, \ k \in \mathbb{Z}$

iff $z = ik, \ k \in \mathbb{Z}$

$h(z) = 0$

iff $z^4 = 1$

iff $z = 1, -1, i, -i$



case 1: $z = \pm 1$

$$g(\pm 1) = \sin\left(\frac{\pi}{i}(\pm 1)\right) \neq 0$$

$$h(\pm 1) = (\pm 1)^4 - 1 = 0$$

$$h'(\pm 1) = 4(\pm 1)^3 \neq 0$$

So we get a pole of order 1 and

$$\operatorname{Res}(f;1) = \frac{g(1)}{h'(1)} = \frac{\sin\left(\frac{\pi}{i}\right)}{4}$$

$$\operatorname{Res}(f;-1) = \frac{g(-1)}{h'(-1)} = \frac{\sin\left(-\frac{\pi}{i}\right)}{-4}$$

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case 2: $z = i$

$$g(i) = \sin\left(\frac{\pi}{i} i\right) = \sin(\pi) = 0$$

$$g'(i) = \cos\left(\frac{\pi}{i} i\right) \cdot \frac{\pi}{i}$$

$$= \cos(\pi) \cdot \frac{\pi}{i} = -\frac{\pi}{i}$$

g has a zero of order 1 at $z_0 = i$

$$\boxed{\begin{array}{l} \text{g's Taylor series at } z_0 = i \text{ is:} \\ g(z) = \left[\underset{\underset{g(i)}{\uparrow}}{0} + \underset{\underset{\frac{g'(i)}{1!}}{\uparrow}}{\frac{-\pi/i}{1}} (z-i) + \dots \right] \end{array}}$$

$$h(z) = z^4 - 1$$

$$h(i) = 0$$

$$h'(i) = 4(i)^3 = -4i \neq 0$$

h has a zero of order 1 at i.

f has a removable singularity at $z_0 = i$ and

$\text{Res}(f; i) = 0$

$$g(z) = \sum_{n=0}^{\infty} \frac{\sin(n) \cdot 5^n}{e^{2n}} (z-1)^n$$

Then,

$$\left| \frac{\sin(n) \cdot 5^n}{e^{2n}} \right| |z-1|^n \leq \frac{5^n}{e^{2n}} |z-1|^n$$

Let's look at

$$\sum_{n=0}^{\infty} \frac{5^n}{e^{2n}} |z-1|^n$$

Ratio time!

$$\lim_{n \to \infty} \left| \frac{5^{n+1}}{e^{2(n+1)}} |z-1|^{n+1} \cdot \frac{e^{2n}}{5^n} \cdot \frac{1}{|z-1|^n} \right|$$

$$= \lim_{n \to \infty} \frac{5}{e^2} |z-1| = \frac{5}{e^2} |z-1|$$

We will get convergence when $\frac{5}{e^2} |z-1| < 1$

or $|z-1| < \frac{e^2}{5}$.

Since $\sum_{n=0}^{\infty} \frac{5^n}{e^{2n}} |z-1|^n$ conoeges

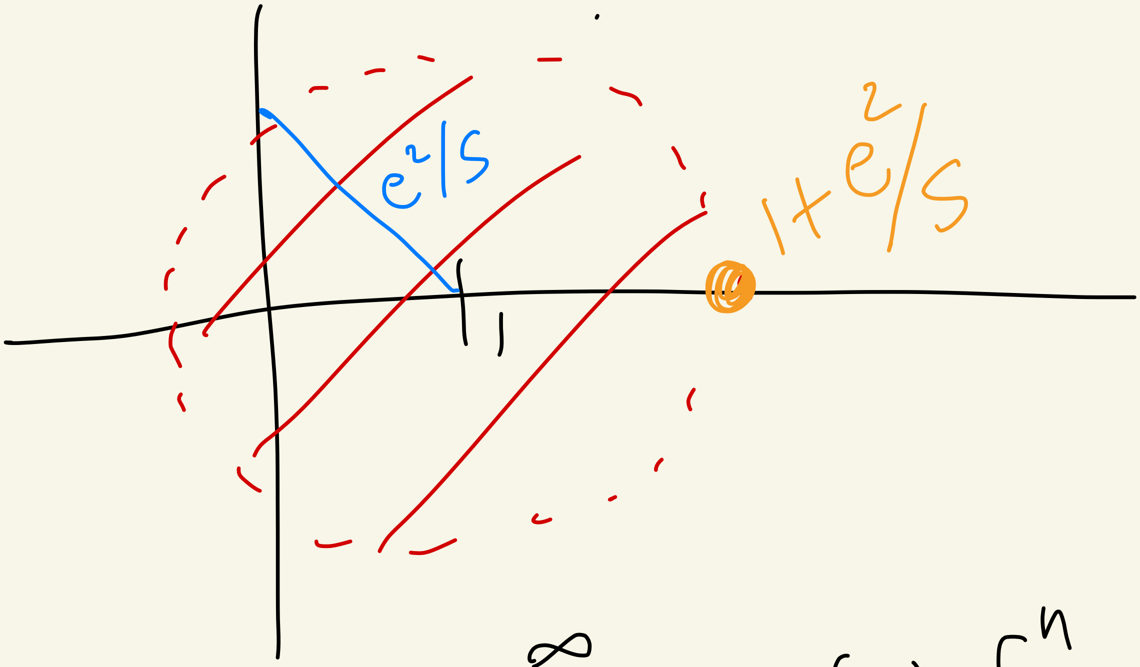
when $|z-1| < \frac{e^2}{5}$

we get by the comparison test

$$g(z) = \sum_{n=0}^{\infty} \frac{\sin(n) \cdot 5^n}{e^{2n}} (z-1)^n$$

converges absolutely when $|z-1| < \frac{e^2}{5}$.

So, the radius of convergence of g is at least $\frac{e^2}{5} \approx 1.4778$



$$g\left(1+\frac{e^2}{5}\right) = \sum_{n=0}^{\infty} \frac{\sin(n)\cdot 5^n}{e^{2n}}\left(1+\frac{e^2}{5}-1\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{\sin(n)\cdot 5^n}{e^{2n}}\left(\frac{e^2}{5}\right)^n$$

$$= \sum_{n=0}^{\infty} \sin(n)$$

And $\lim_{n\to\infty} \sin(n)$ DNE

By divergence thm, g's sum

doesn't converge at $1+\frac{e^2}{5}$

So, $R=\frac{e^2}{5}$ is y's radius of convergence.