Math 5680

5/8/23

HW 7 - Identity theorem will not be on the final contrary to the study guide

HW 6

(1)

$$\int_0^{2\pi} \frac{d\theta}{2-\sin(\theta)}$$

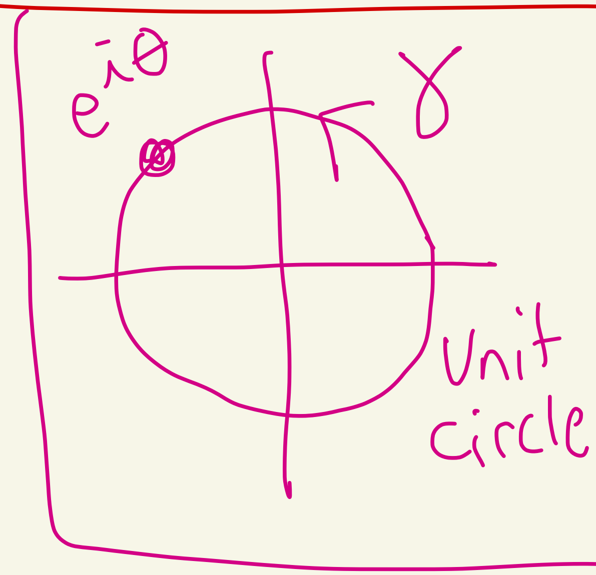
$$z = e^{i\theta}, \quad 0 \leq \theta \leq 2\pi$$

$$\sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{z - \frac{1}{z}}{2i}$$

$$dz = ie^{i\theta}\, d\theta$$

$$d\theta = \frac{1}{i} \frac{1}{e^{i\theta}} dz = -i \frac{1}{z} dz$$

$$= \int_\gamma \frac{-i \frac{1}{z} dz}{2 - \left(\frac{z - \frac{1}{z}}{2i} \right)}$$



$$= \int_\gamma \frac{2\,\frac{1}{z}}{4i - z + \frac{1}{z}}\,dz$$

$\boxed{x\,2\,i}$

$$= \int_\gamma \frac{z}{-z^2+4iz+1}\,dz$$

$-z^2+4iz+1=0$ iff

$$z = \frac{-4i \pm \sqrt{(4i)^2-4(-1)(1)}}{2(-1)}$$

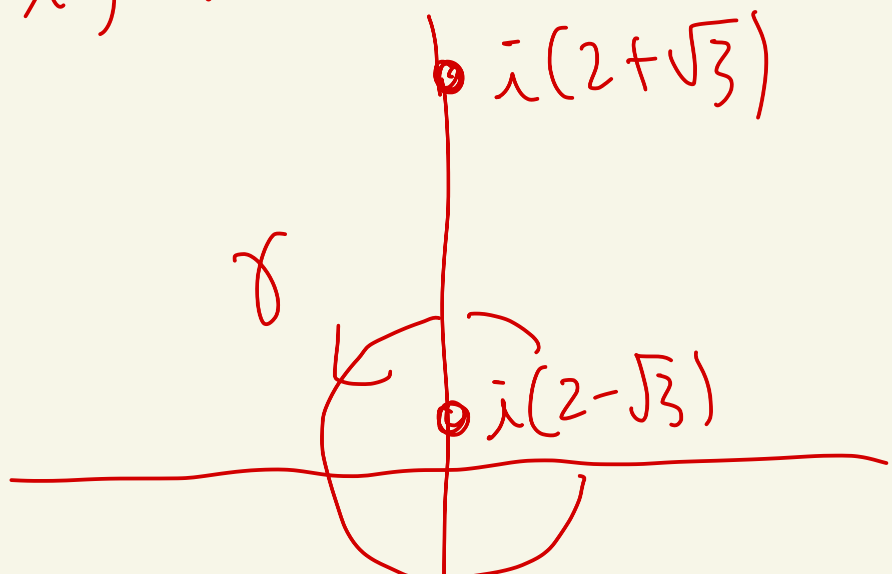
$$= \frac{-4i \pm \sqrt{-16+4}}{-2}$$

$$= \frac{-4i \pm \sqrt{-12}}{-2} = \frac{-4i \pm \sqrt{12}\,i}{-2}$$

$$\frac{-4i \pm 2\sqrt{3}i}{-2} = 2i \pm \sqrt{3}i$$

$$= i(2+\sqrt{3}),\ i(2-\sqrt{3})$$

$$\approx (3.732)i,\ (0.2679)i$$



f

$$\int_\gamma \frac{z}{-z^2+4iz+1}\,dz = 2\pi i \operatorname{Res}\left(f; i(2-\sqrt{3})\right)$$

Note

$$\frac{z}{-z^2+4iz+1} = \frac{z}{-\left[z-i(2+\sqrt{3})\right]\left[z-i(2-\sqrt{3})\right]}$$

$$= \frac{\overbrace{-z/\left[z-i(2+\sqrt{3})\right]}^{\varphi(z)}}{\left[z-i(2-\sqrt{3})\right]}$$

So we have a simple pole at $i(2-\sqrt{3})$ and

$$\operatorname{Res}(f; i(2-\sqrt{3})) = \frac{\varphi^{(1-1)}(i(2-\sqrt{3}))}{(1-1)!}$$

$$= \varphi(i(2-\sqrt{3}))$$

$$= \frac{-2}{\left[i(2-\sqrt{3}) - i(2+\sqrt{3})\right]} = \frac{-2}{-2i\sqrt{3}}$$

$$= \frac{1}{i\sqrt{3}} \cdot \frac{-i\sqrt{3}}{-i\sqrt{3}} = -\frac{\sqrt{3}}{3} i$$

So,

$$\int_0^{2\pi} \frac{d\theta}{2-\sin(\theta)} = 2\pi i \left(-\frac{\sqrt{3}}{3} i\right)$$

$$= \frac{2\sqrt{3}}{3} \pi$$

$\boxed{\text{HW 6}}$

(4) $\displaystyle\int_0^\infty \underbrace{\frac{1+x^2}{1+x^4}}_{f} dx$

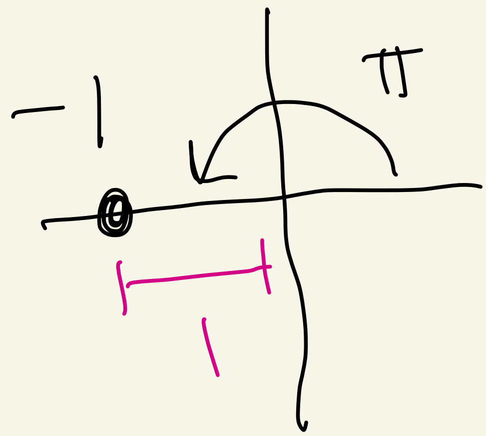
f is even since $f(-x) = f(x)$

So, $\displaystyle\int_0^\infty \frac{1+x^2}{1+x^4} dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{1+x^2}{1+x^4} dx$

$\displaystyle = \frac{1}{2} \lim_{R\to\infty} \underbrace{\int_{-R}^{R} \frac{1+x^2}{1+x^4} dx}_{\text{Cauchy principal value}}$

Where are the singularities of $f(z) = \dfrac{1+z^2}{1+z^4}$?

When $z^4 = -1$

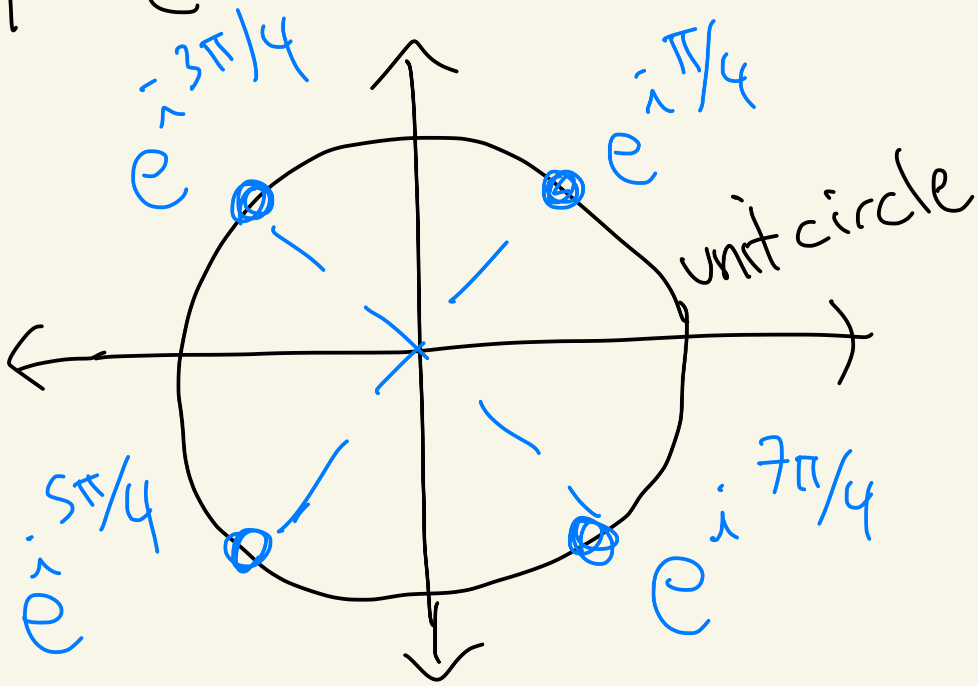


or

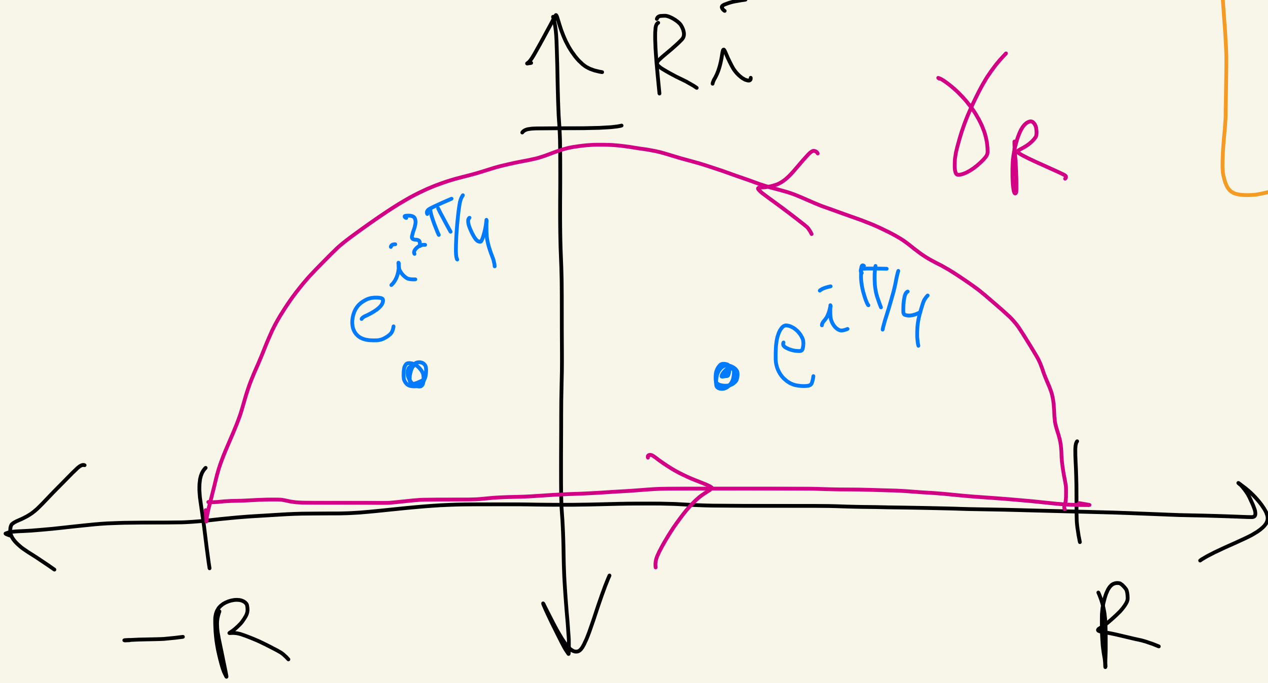
$$z^4 = 1 \cdot e^{i\pi}$$

Solutions:

$$z_k = 1^{1/4}\, e^{i\left(\frac{\pi}{4} + \frac{2\pi k}{4}\right)} \qquad k = 0,1,2,3$$



Let $R > 1$.



Note

$$\int_{\gamma_R} \frac{1+z^2}{1+z^4}\,dz = 2\pi i\left[\operatorname{Res}\left(f; e^{i\pi/4}\right) + \operatorname{Res}\left(f; e^{i\frac{3\pi}{4}}\right)\right]$$

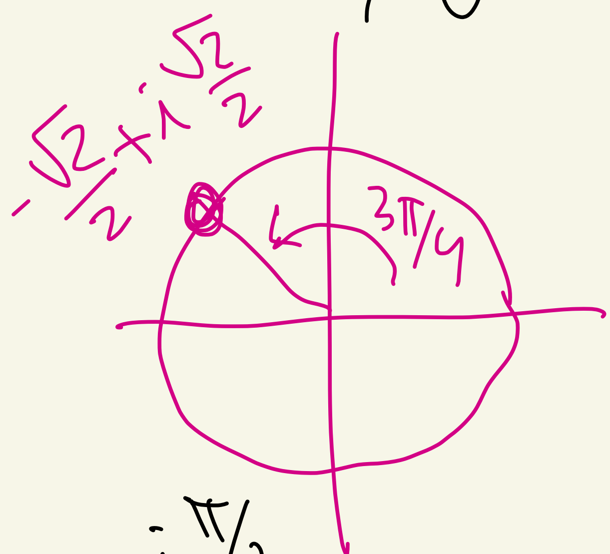
Let $\dfrac{1+z^2}{1+z^4} = \dfrac{g(z)}{h(z)}$ $\quad \boxed{h'(z) = 4z^3}$

Note $g(e^{i\pi/4}) = 1 + e^{i\pi/2} = 1 + i \neq 0$

and $h(e^{i\pi/4}) = 0$

and $h'(e^{i\pi/4}) = 4e^{i3\pi/4} = 4\left[-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right] \neq 0$

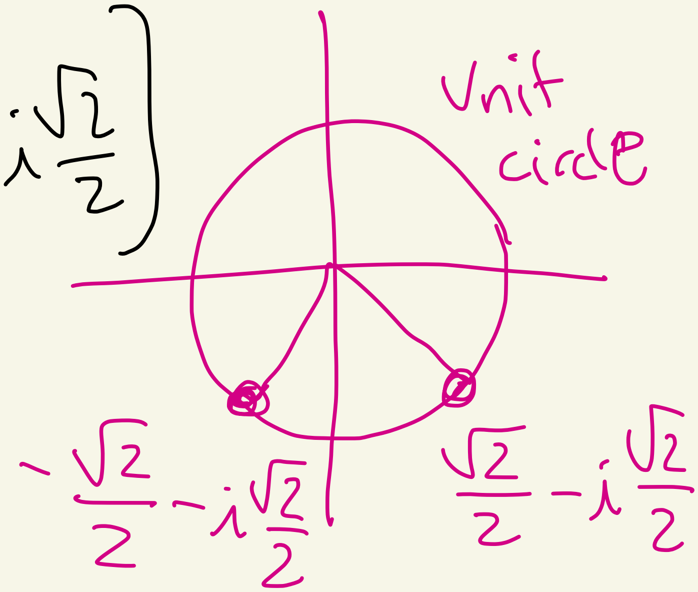
So we have a simple pole and



$$\mathrm{Res}(f; e^{i\pi/4}) = \frac{g(e^{i\pi/4})}{h'(e^{i\pi/4})} = \frac{1+e^{i\pi/2}}{4e^{i3\pi/4}}$$

$$= \frac{1}{4}\left[e^{-i\frac{3\pi}{4}} + e^{-i\pi/4}\right]$$

$$= \frac{1}{4}\left[-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right]$$



$$= \frac{1}{4}\left[-\sqrt{2}\, i\right]$$

$$= \frac{-\sqrt{2}}{4} i$$

$$g\left(e^{i\frac{3\pi}{4}}\right) = 1 + e^{i\frac{3\pi}{2}} = 1 - i \neq 0$$

$$h\left(e^{i3\pi/4}\right) = 0$$

$$h'\left(e^{i3\pi/4}\right) = 4e^{i9\pi/4} = 4e^{i\pi/4} \neq 0$$

Simple pole and

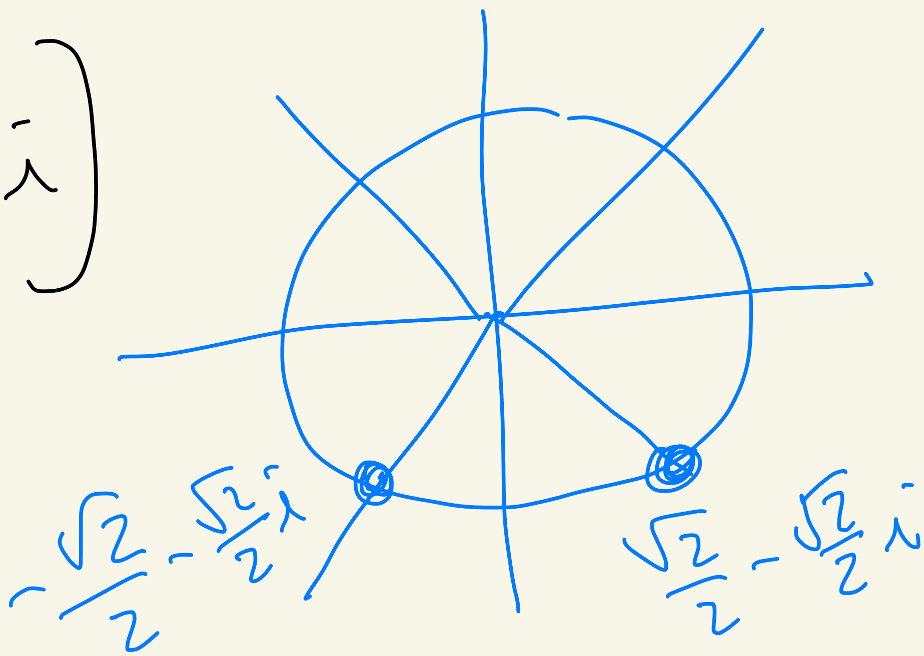
$$\mathrm{Res}\left(f; e^{i3\pi/4}\right) = \frac{1+e^{i3\pi/2}}{4e^{i\pi/4}}$$

$$= \frac{1}{4}\left[e^{-i\pi/4} + e^{i5\pi/4}\right]$$

$$= \frac{1}{4}\left[\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i - \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right]$$

$$= \frac{1}{4}\left[-\sqrt{2}\,i\right]$$

$$= \boxed{-\frac{\sqrt{2}}{4}i}$$



Thus,

$$\int_{\gamma_R} \frac{1+z^2}{1+z^4}\,dz = 2\pi i\left[-\frac{\sqrt{2}}{4}i - \frac{\sqrt{2}}{4}i\right]$$

$$= 2\pi i\left[-\frac{\sqrt{2}}{2}i\right]$$

$$= \boxed{\sqrt{2}\pi}$$

So,

show $\to 0$

$$\lim_{R\to\infty}\int_{-R}^{R} \frac{1+x^2}{1+x^4}\,dx + \lim_{R\to\infty}\int_{C_R} \frac{1+z^2}{1+z^4}\,dz$$

$$= \sqrt{2}\pi$$

If $z \in C_R$, then $|z| = R$.

And

$$|1+z^2| \le |1| + |z|^2 = 1+R^2$$

$$|1+z^4| \ge \big|\, |1| - |z|^4 \big| = |1-R^4| = R^4 - 1$$

$R > 1$
$R^4 > 1$
$R^4 - 1 > 0$
$1 - R^4 < 0$

So,

$$\left| \int_{C_R} \frac{1+z^2}{1+z^4}\, dz \right| \le \frac{1+R^2}{R^4-1} \cdot \underbrace{\pi R}_{\text{arclength of } C_R}$$

$$= \frac{\pi R + \pi R^3}{R^4 - 1} \longrightarrow 0$$

as $R \to \infty$

So, $\displaystyle\int_{-\infty}^{\infty} \frac{1+x^2}{1+x^4}\,dx + 0 = \sqrt{2}\,\pi$

Thus, $\displaystyle\int_{0}^{\infty} \frac{1+x^2}{1+x^4}\,dx = \frac{\sqrt{2}}{2}\pi.$ $\blacksquare$