1

(a) $\displaystyle\sum_{n=7}^{\infty} \frac{\pi^2-10}{5^{n+1}e^{n-1}} = (\pi^2-10)\sum_{n=7}^{\infty}\frac{1}{5^{n+1}e^{n-1}}$

$$= (\pi^2-10)\left[\frac{1}{5^8e^6}+\frac{1}{5^9e^7}+\frac{1}{5^{10}e^8}+\dots\right]$$

$$= (\pi^2-10)\frac{1}{5^8e^6}\left[1+\frac{1}{5e}+\left(\frac{1}{5e}\right)^2+\left(\frac{1}{5e}\right)^3+\dots\right]$$

$$= \boxed{(\pi^2-10)\frac{1}{5^8e^6}\frac{1}{1-\frac{1}{5e}}}$$

converges since $\left|\frac{1}{5e}\right|<1$

(b) $\displaystyle\left|\frac{5+(-1)^n}{n^3+1}\right| = \frac{|5+(-1)^n|}{n^3+1} < \frac{|5+(-1)^n|}{n^3} \le \frac{5+1}{n^3} = \frac{6}{n^3}$

Since $\displaystyle\sum_{n=1}^{\infty}\frac{6}{n^3} = 6\sum_{n=1}^{\infty}\frac{1}{n^3}$ converges, by the comparison test so does $\displaystyle\sum_{n=1}^{\infty}\left|\frac{5+(-1)^n}{n^3+1}\right|$.

Thus, the series converges absolutely.

② Let $\varepsilon > 0$.

Note that

$$\left|z_n - 0\right| = \left|\frac{5}{n} + i\frac{5}{n^6}\right| \leq \left|\frac{5}{n}\right| + \left|i\frac{5}{n^6}\right|$$

$$= \left|\frac{5}{n}\right| + |i|\left|\frac{5}{n^6}\right|$$

$$= \frac{5}{n} + \frac{5}{n^6}$$

$$\leq \frac{5}{n} + \frac{5}{n}$$

$$= \frac{10}{n}$$

Note that $\frac{10}{n} < \varepsilon$ iff $\frac{10}{\varepsilon} < n$.

Let $N > \frac{10}{\varepsilon}$.

Then if $n \geq N$, then $|z_n - 0| \leq \frac{10}{n} < \varepsilon$.

Thus, $\lim_{n\to\infty} z_n = 0$.

③ Let $\varepsilon > 0$.

Let $z \in D$.

Then, $|z-1| \leq 10$.

Also,

$$|f_n(z) - 0| = \left|\frac{(z-1)^4}{n^5}\right| = \frac{|z-1|^4}{n^5} \leq \frac{10^4}{n^5}.$$

Note that $\frac{10^4}{n^5} < \varepsilon$ if $\frac{10^4}{\varepsilon} < n^5$ if $\sqrt[5]{\frac{10^4}{\varepsilon}} < n$.

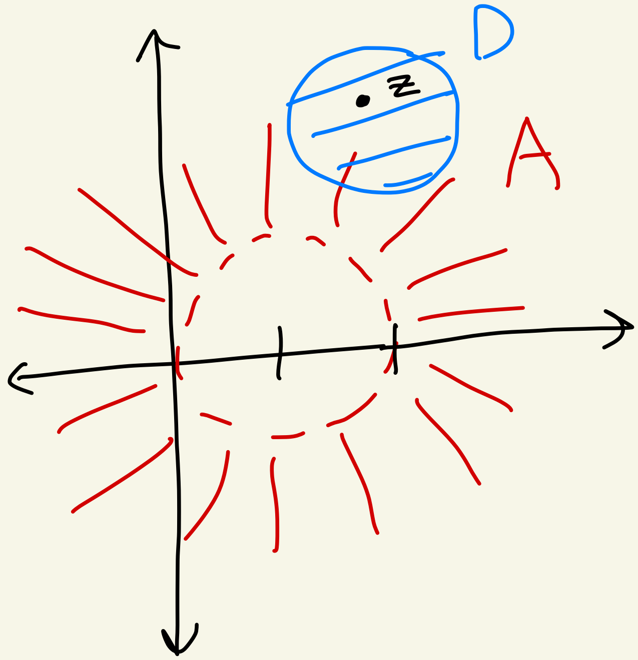
Let $N > \sqrt[5]{\frac{10^4}{\varepsilon}}$.

If $n \geq N$ and $z \in D$, then

$$|f_n(z) - 0| \leq \frac{10^4}{n^5} < \varepsilon.$$

Thus, f_n converges to the zero function uniformly on D.

④ Let D be a closed disc in A.



If $z \in D$, then $|z-1| > 1$ and so

$$\left| \frac{1}{n^2(z-1)^n} \right| < \frac{1}{n^2}$$

$|z-1| > 1$
so, $\frac{1}{|z-1|} < 1$

Set $M_n = \frac{1}{n^2}$.

Note that $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

By the Weierstrass M-test, $\sum_{n=1}^{\infty} \frac{1}{n^2(z-1)^n}$ converges uniformly on D.

Thus, by the analytic convergence theorem $\sum_{n=1}^{\infty} \frac{1}{n^2(z-1)^n}$ is analytic on A.

5 This is HW 0 problem 3.

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6 This is HW 1 Problem 2.