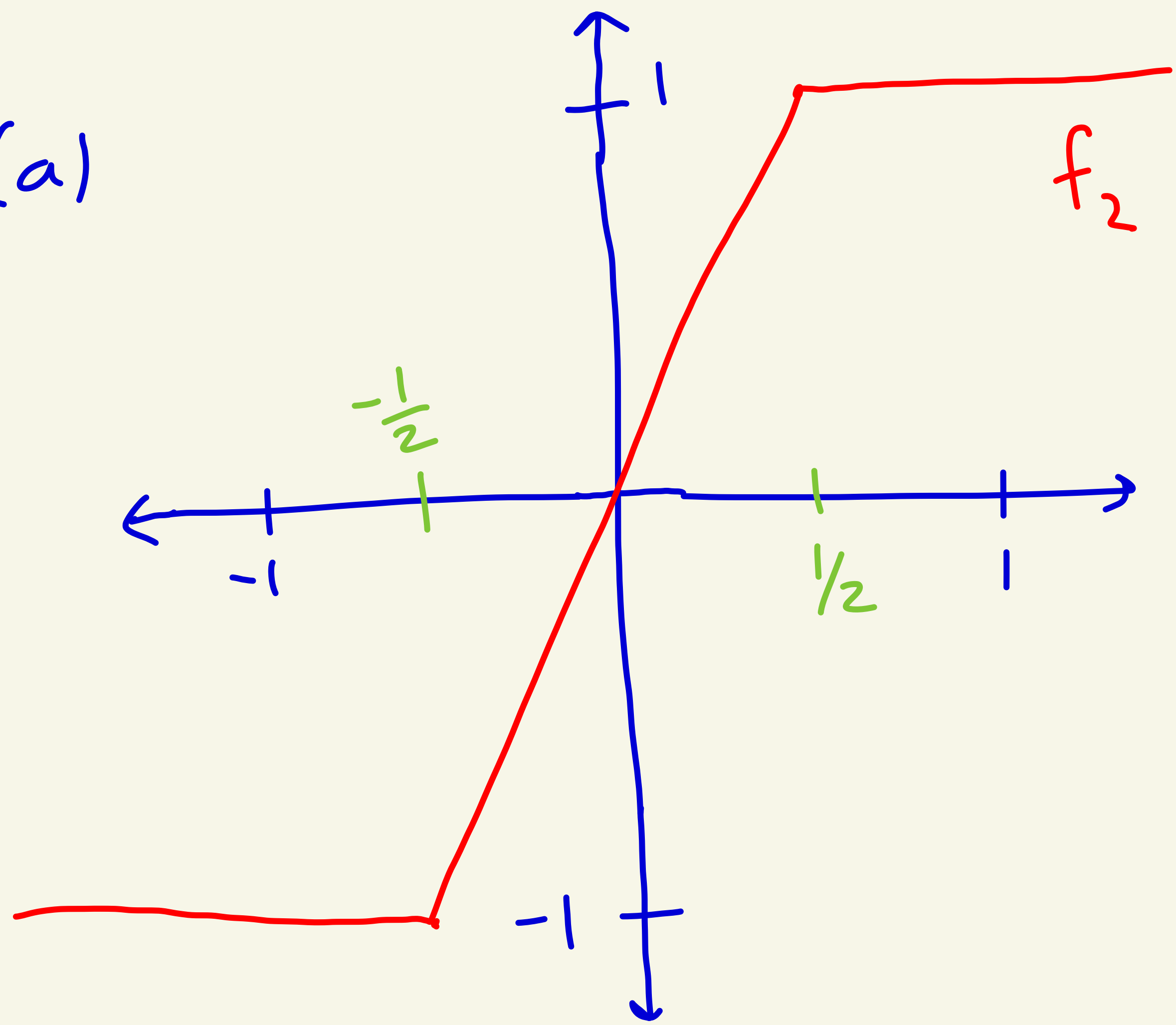
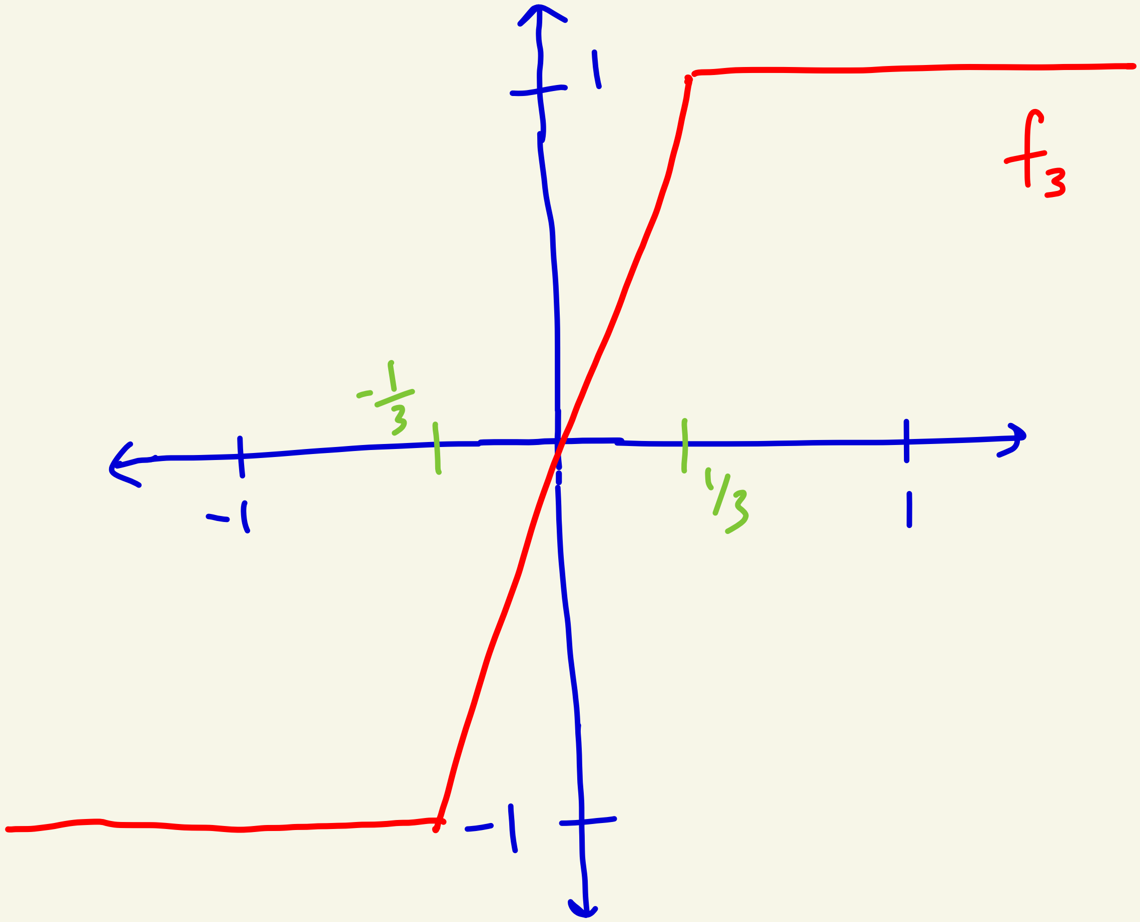
5680
HW 2
Solutions

① We have that $A = \mathbb{R} \subseteq \mathbb{C}$ and for $n \geq 2$ we have $f_n : \mathbb{R} \to \mathbb{C}$ given by

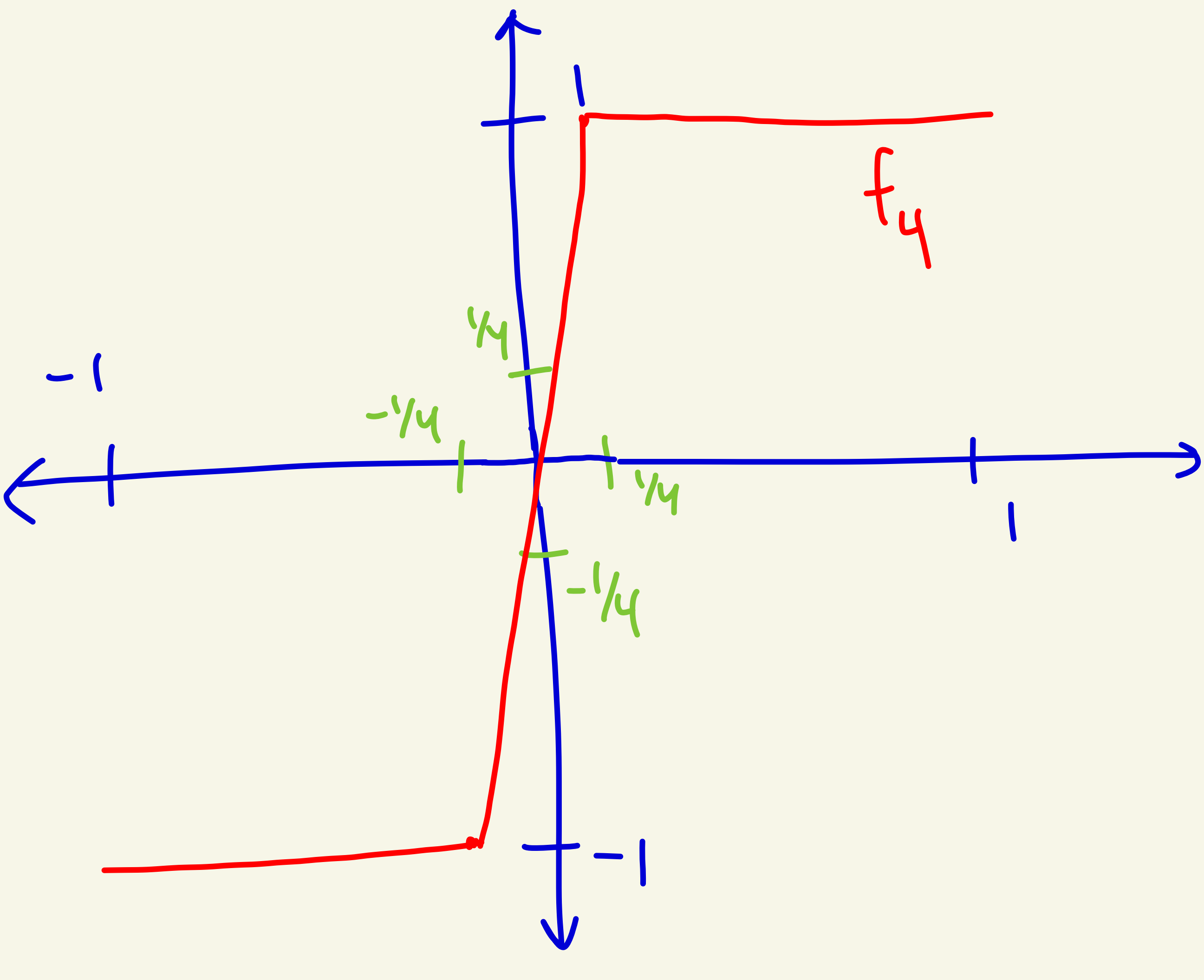
$$f_n(x) = \begin{cases} -1 & \text{for } x \leq -\frac{1}{n} \\ nx & \text{for } -\frac{1}{n} < x < \frac{1}{n} \\ 1 & \text{for } \frac{1}{n} \leq x \end{cases}$$

(a)





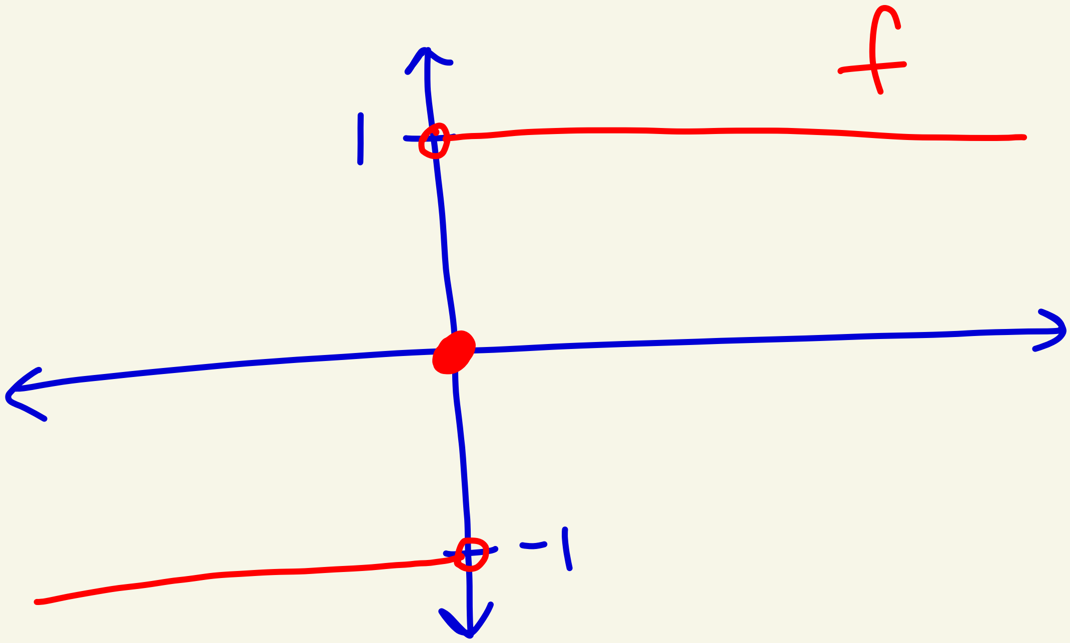
1
-1
$-\frac{1}{3}$
$1/3$
1
-1
f_3



1
f_4
$1/4$
$-1/4$
-1
$1/4$
1
$-1/4$
-1

(b) Let

$$f(x) = \begin{cases} -1 & \text{for } x < 0 \\ 0 & \text{for } x = 0 \\ 1 & \text{for } x > 1 \end{cases}$$



Show that $f_n(x)$ converges to $f(x)$ pointwise on $\mathbb{R}$

Let $x \in \mathbb{R}$.

<u>case 1: Suppose that $x = 0$</u>.

Then

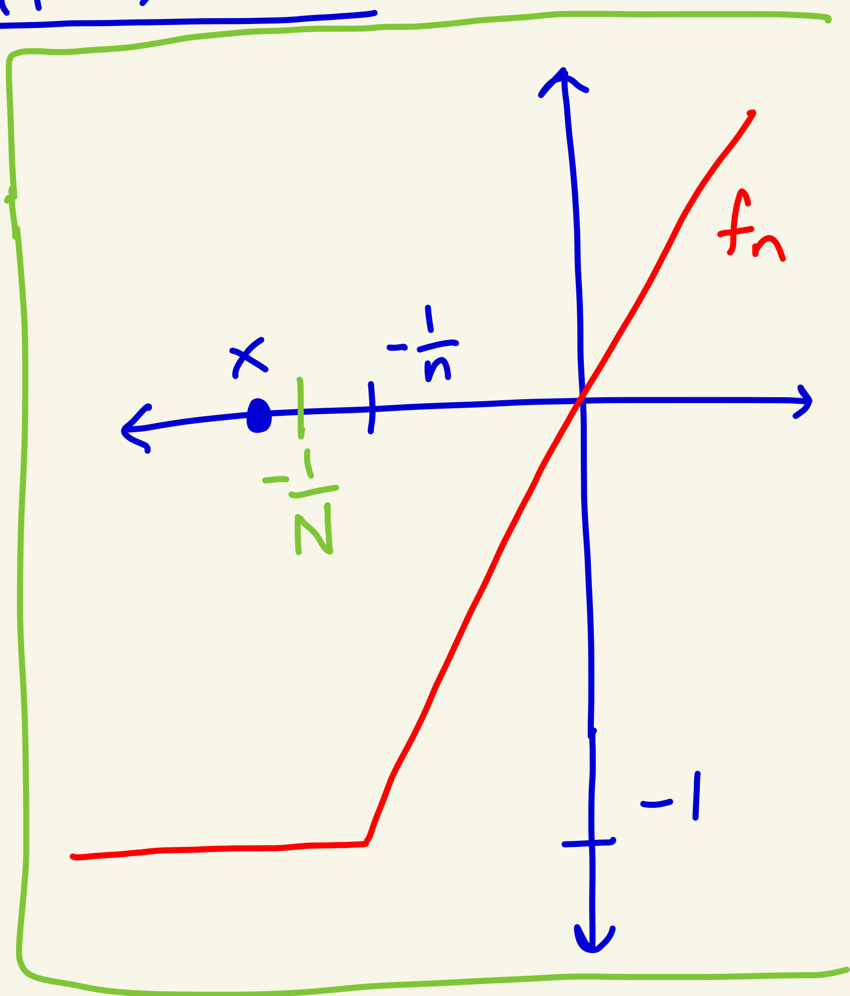
$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} f_n(0) = \lim_{n\to\infty} 0 = 0 = f(0).$$

So, $f_n(0)$ converges to $f(0)$.

Case 2: Suppose that $x < 0$.

Let $\varepsilon > 0$.
Pick an $N \geq 1$ where

$$x < -\frac{1}{N}$$



Then if $n \geq N$ we have that

$$x < -\frac{1}{N} \leq \frac{-1}{n}$$

and so $f_n(x) = -1$ for $n \geq N$.

Thus, if $n \geq N$ then

$$|f_n(x) - f(x)| = |-1-(-1)| = 0 < \varepsilon.$$

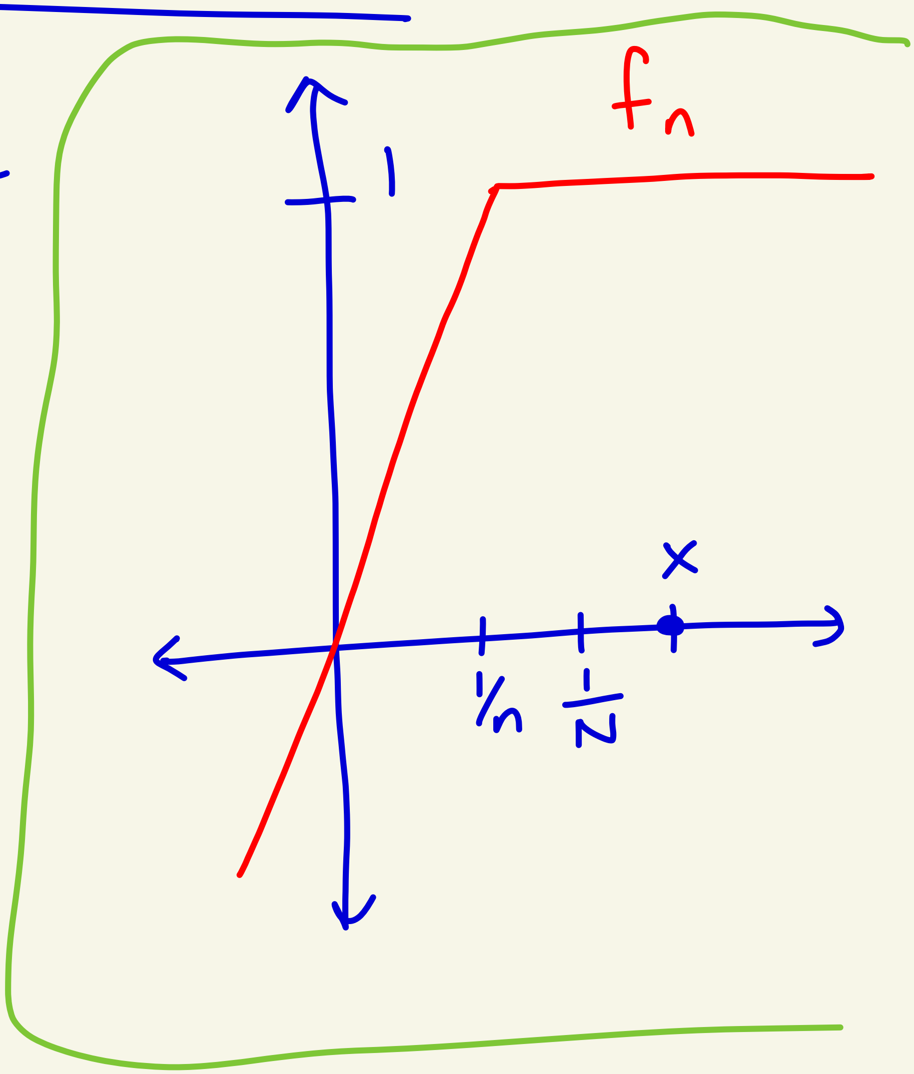
So, $\lim_{n\to\infty} f_n(x) = -1 = f(x)$.

Case 3: Suppose that $x > 0$.

Let $\varepsilon > 0$.
Pick an $N \geq 1$ where
$\frac{1}{N} < x$.

Then if $n \geq N$,
then
$$\frac{1}{n} \leq \frac{1}{N} < x.$$

And so, if
$n \geq N$, then



$f_n(x) = 1$.
Thus, if $n \geq N$ then
$$|f_n(x) - f(x)| = |1 - 1| = 0 < \varepsilon$$
So, $\lim_{n\to\infty} f_n(x) = 1 = f(x)$.

Combining the three cases we see that for any fixed $x \in \mathbb{R}$ we have that

$$\lim_{n \to \infty} f_n(x) = f(x)$$

Thus, f_n converges pointwise to f on $A = \mathbb{R}$.

② (Method 1 - by definition)

Let $f_0(z) = 0$ for all $z \in D(0;r)$.

Let $\varepsilon > 0$.

Then, if $z \in D(0;r)$ then

$$|f_n(z) - f_0(z)| = \left| \frac{z^3}{n^2} - 0 \right|$$

$$= \left| \frac{z^3}{n^2} \right| = \frac{|z|^3}{n^2} \underset{z \in D(0;r)}{<} \frac{r^3}{n^2}$$

We need $\frac{r^3}{n^2} < \varepsilon$.

Note that $\frac{r^3}{n^2} < \varepsilon$ iff $\frac{r^3}{\varepsilon} < n^2$

iff $\sqrt{\frac{r^3}{\varepsilon}} < n$.

Let $N > \sqrt{\frac{r^3}{\varepsilon}}$.

Then if $n \geq N > \sqrt{\frac{r^3}{\varepsilon}}$ we have

$$|f_n(z) - f_0(z)| < \frac{r^3}{n^2} < \varepsilon$$

for all $z \in D(0;r)$.

Thus, $f_n \to f_0$

uniformly on $D(0;r)$.

Method 2

Let $z \in D(0;r)$.

Then $|z| < r$.

So, $|f_n(z)| = \left|\frac{z^3}{n^2}\right| = \frac{|z|^3}{n^2} < \frac{r^3}{n^2}$

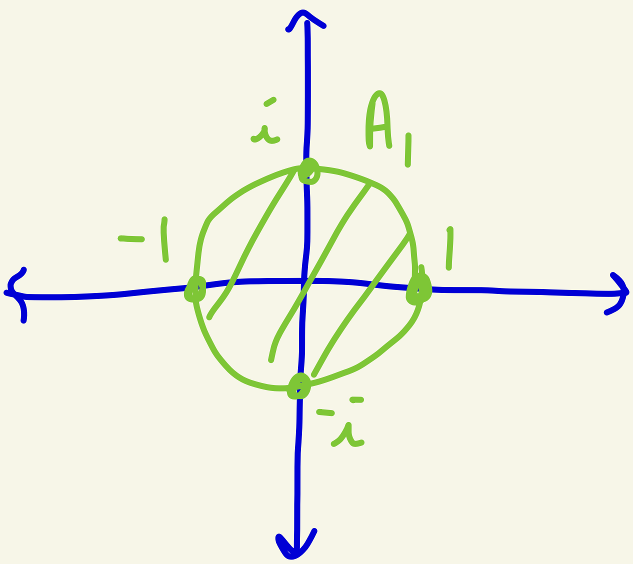
Let $M_n = \frac{r^3}{n^2}$

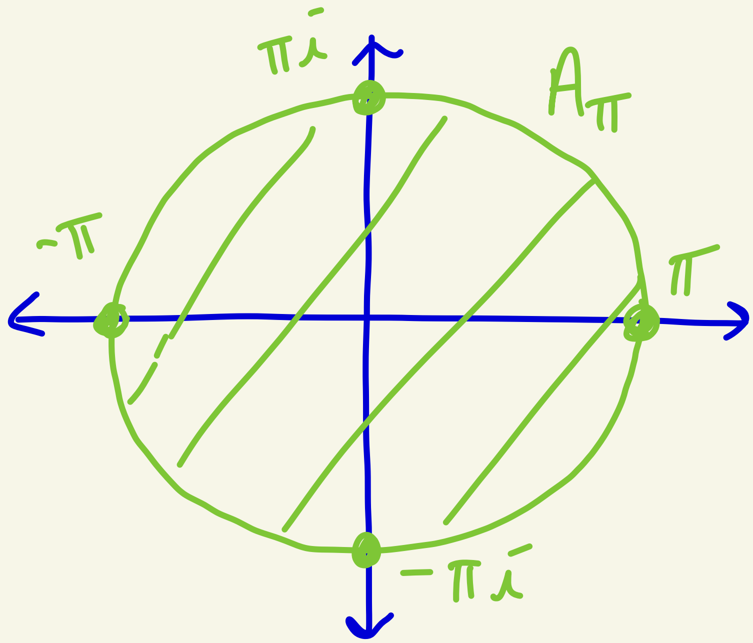
Then $\sum_{n=1}^{\infty} M_n = r^3 \sum_{n=1}^{\infty} \frac{1}{n^2}$ which converges since its a $p=2$ series.

By the Weierstrass M-Test, $\sum_{n=1}^{\infty} f_n(z)$ converges uniformly on $D(0;r)$.

By HW 2 #6, the sequence $(f_n)_{n=1}^{\infty}$ converges uniformly to the zero function on $D(0;r)$.

③ (a)

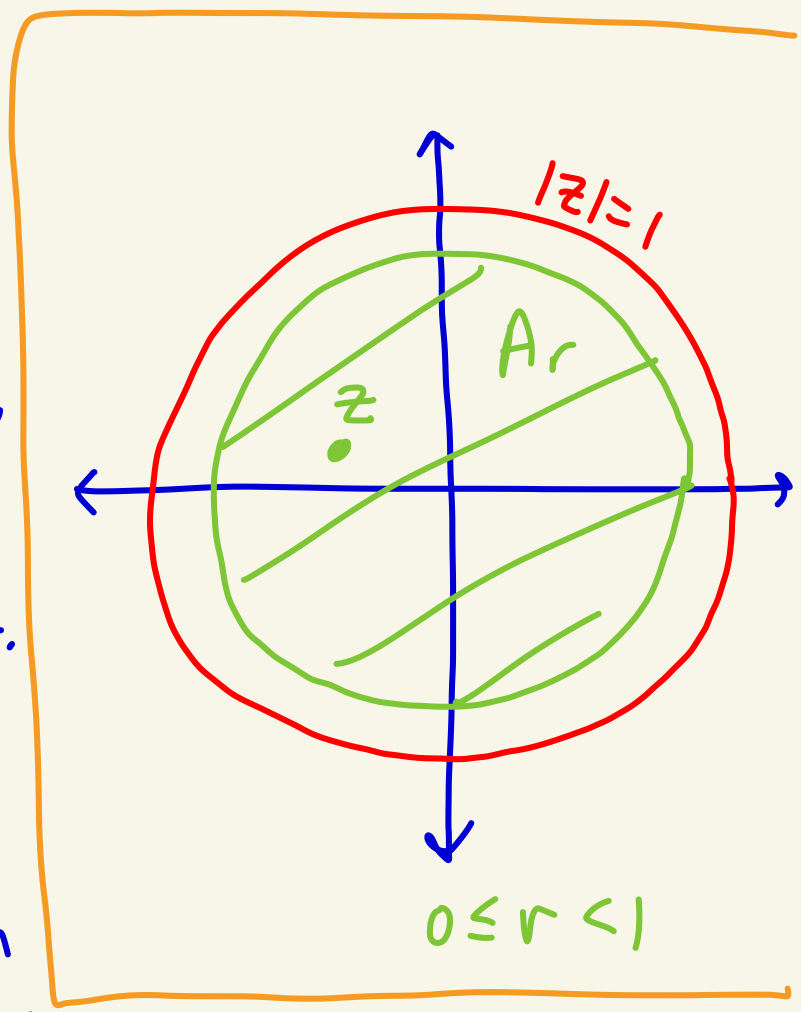




(b) Let $0 \le r < 1$.

Let $g_n(z) = \frac{z^n}{n}$, so that $\sum_{n=1}^{\infty} \frac{z^n}{n} = \sum_{n=1}^{\infty} g_n(z)$.

Let's use the Weierstrass M-Test.



ⓘ Let $z \in A_r$

Then $|z| \le r$ and so

$$|g_n(z)| = \left| \frac{z^n}{n} \right| = \frac{|z|^n}{n} \le \frac{r^n}{n}.$$

Let $M_n = \frac{r^n}{n}$.

Then, $|g_n(z)| \leq M_n$ for all $z \in A_r$.

(ii) Note that $\frac{r^n}{n} \leq r^n$ for $n \geq 1$.

Also, since $0 \leq r < 1$, the geometric series $\sum_{n=1}^{\infty} r^n$ converges.

Thus, by the comparison test,

$$\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{r^n}{n}$$

converges.

Thus, the conditions (i) & (ii) of the Weierstrass M-Test hold on A_r.

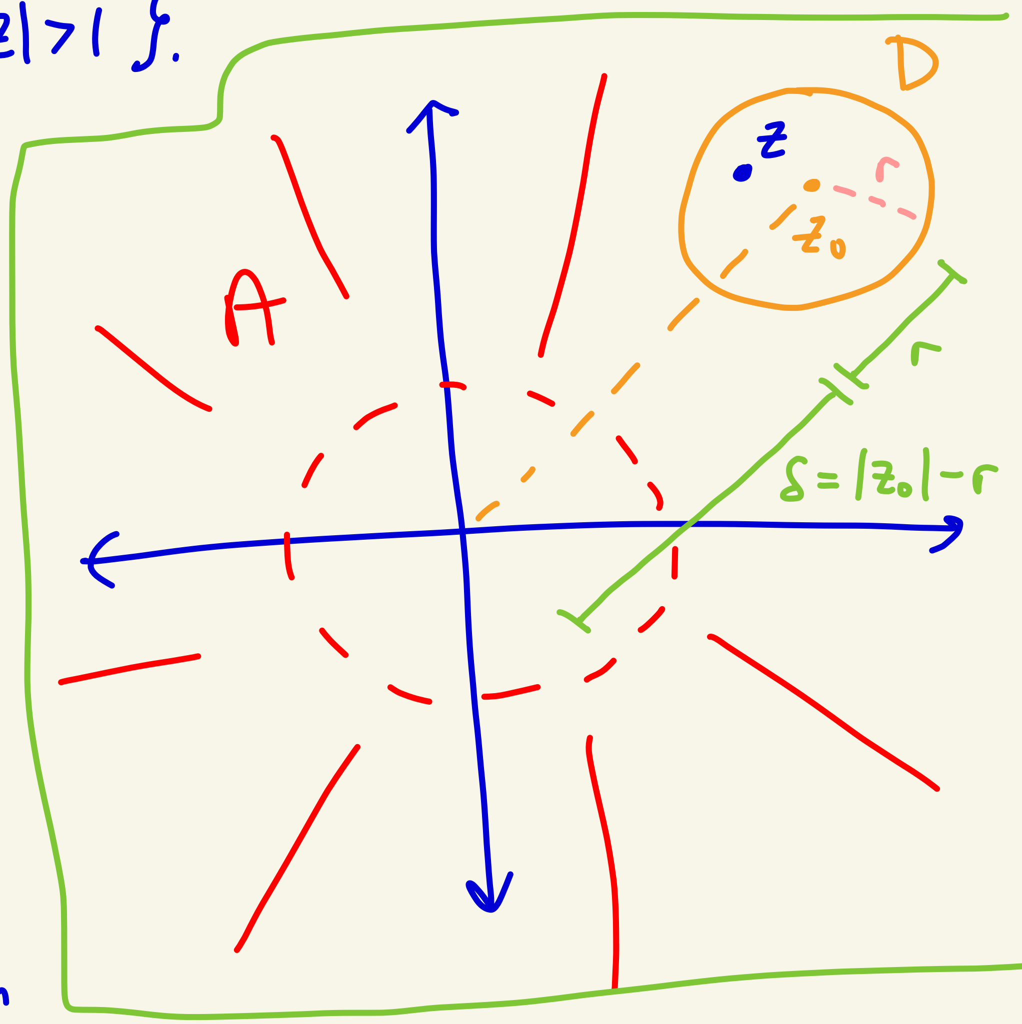
So, $\sum_{n=1}^{\infty} g_n(z) = \sum_{n=1}^{\infty} \frac{z^n}{n}$ converges absolutely and uniformly on A_r where $0 \leq r < 1$.

④ We use the analytic convergence theorem.

Let $A = \{ z \mid |z| > 1 \}$.

Let D be a closed disk in A.

We will show that $\sum_{n=1}^{\infty} \frac{1}{z^n}$ converges uniformly on D.



We do this with the Weierstrass M-test.

Let $g_n(z) = \frac{1}{z^n}$ for $n \geq 1$.

Let z_0 be the center of D and r be the radius of D. So, $D = \{ z \mid |z - z_0| \leq r \}$.

Let $\delta = |z_0| - r$. [See picture]

Note that $\delta > 1$ because D is in A.

<u>Claim:</u> If $z \in D$, then $|z| \geq \delta$.

<u>proof of claim:</u> Let $z \in D$. Then, $|z - z_0| \leq r$.

Thus,

$$
\begin{aligned}
|z_0| &= |z_0 - z + z| \\
&\leq |z_0 - z| + |z| \\
&= |(-1)(z - z_0)| + |z| \\
&= |-1||z - z_0| + |z| \\
&= |z - z_0| + |z| \\
&\leq r + |z|
\end{aligned}
$$

Thus, $|z_0| \leq r + |z|$.

So, $|z| \geq |z_0| - r = \delta$.

Therefore, if $z \in D$, then $|z| \geq \delta$.

$\boxed{\text{Claim}}$

So, if $z \in D$, then

$$|g_n(z)| = \left|\frac{1}{z^n}\right| = \frac{1}{|z|^n} \leq \frac{1}{\delta^n} = \left(\frac{1}{\delta}\right)^n$$

Let $M_n = \left(\frac{1}{\delta}\right)^n$.

So, if $z \in D$, then $|g_n(z)| \leq M_n$.

Also, since $\delta > 1$ we know $\frac{1}{\delta} < 1$.

So, $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \left(\frac{1}{\delta}\right)^n$ is a convergent geometric series.

Thus, by the Weierstrass M-Test,

$\sum_{n=1}^{\infty} g_n(z) = \sum_{n=1}^{\infty} \frac{1}{z^n}$ converges uniformly on D.

So, by the analytic convergence theorem

$g(z)$ is an analytic function on A.

(b) When $z \in A$ we have that

$$g'(z) = \sum_{n=1}^{\infty} g_n'(z) = \sum_{n=1}^{\infty} \left(z^{-n}\right)' = \sum_{n=1}^{\infty} -n z^{-n-1}$$

$$= -\sum_{n=1}^{\infty} \frac{n}{z^{n+1}}$$

5

Let $g(z) = \sum_{n=1}^{\infty} \frac{1}{n! z^n}$

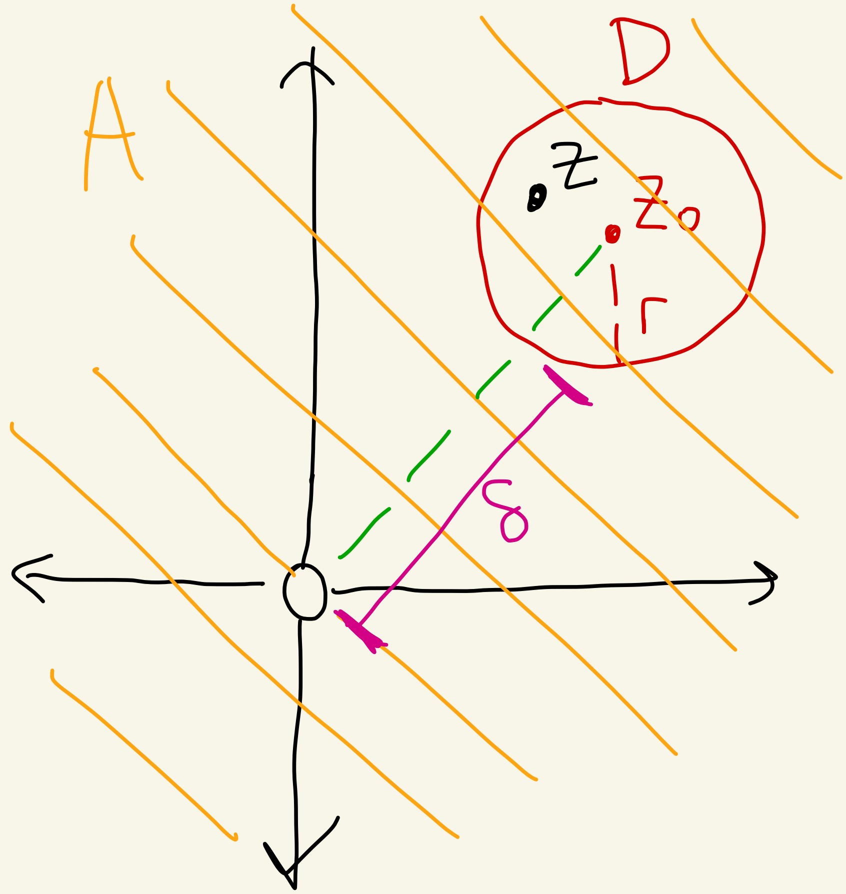
$A = \mathbb{C} - \{0\}$

(a) Show g is analytic on A
(b) Find a formula for g' on A.

<u>proof</u>: We use the analytic convergence theorem.
Let D be a closed disc in A.
Let D have center z_0 and radius r.
So, $D = \{z \mid |z - z_0| \leq r\}$

Let

$\delta = |z_0| - r > 0$



<u>Claim:</u> If $z \in D$, then $|z| \geq \delta$.

<u>pf of claim:</u> Let $z \in D$.

Then, $|z - z_0| \leq r$.

Thus,

$$
\begin{aligned}
|z_0| &= |z_0 - z + z| \\
&\leq |z_0 - z| + |z| \\
&= |z - z_0| + |z| \\
&= r + |z|.
\end{aligned}
$$

So, $\underbrace{|z_0| - r}_{\delta} \leq |z|$.

Thus, $\delta \leq |z|$. $\boxed{\text{claim}}$

$\boxed{|z| \geq \delta}$

Thus, if $z \in D$, then

$$\left| \frac{1}{n!} \cdot \frac{1}{z^n} \right| = \frac{1}{n!} \cdot \frac{1}{|z|^n} \leq \underbrace{\frac{1}{n!} \frac{1}{\delta^n}}_{M_n}$$

Let $M_n = \frac{1}{n!} \frac{1}{\delta^n}$.

Does $\sum_{n=1}^{\infty} M_n$ converge?

Let's use the ratio test

$$\lim_{n\to\infty}\left|\frac{M_{n+1}}{M_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{1}{(n+1)!}\,\frac{1}{\delta^{n+1}}}{\frac{1}{n!}\,\frac{1}{\delta^{n}}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{n!}{(n+1)!}\cdot\frac{\delta^{n}}{\delta^{n+1}}\right| = \lim_{n\to\infty}\left|\frac{1}{(n+1)\delta}\right|$$

$$\boxed{(n+1)! = (n+1)\cdot[n!]}$$

$$= 0 \leftarrow \boxed{r}$$

Since $0<1$, by the ratio test $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty}\frac{1}{n!}\frac{1}{\delta^n}$ converges

By the Weierstrass M-test the series $\sum_{n=1}^{\infty}\frac{1}{n!}\frac{1}{z^n}$ converges uniformly (and absolutely) on D.

By the analytic convergence theorem

(a) $g(z) = \sum_{n=1}^{\infty} \frac{1}{n!} \cdot \frac{1}{z^n}$ is

analytic on A,

and

(b) if $z \in A$, then

$$g'(z) = \sum_{n=1}^{\infty} \left(\frac{1}{n!} \frac{1}{z^n} \right)'$$

$$\boxed{\begin{aligned} &(z^{-n})' \\ &= -n z^{-n-1} \end{aligned}}$$

$$= \sum_{n=1}^{\infty} \frac{-n}{n!} \cdot \frac{1}{z^{n+1}}$$

$$= - \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \frac{1}{z^{n+1}}$$

⑥ (Method 1)

Suppose that $\sum_{k=1}^{\infty} g_k(z)$ converges uniformly on some subset $A \subseteq \mathbb{C}$.

Let $\varepsilon > 0$.

By the Cauchy criterion, there exists $N > 0$ so that if $n \geq N$ and $z \in A$ then

$$\left| \sum_{k=n+1}^{n+p} g_k(z) \right| < \varepsilon$$

for $p = 1, 2, 3, 4, \ldots$

Take $p = 1$ to get that if $m \geq N$ and $z \in A$ then $|g_{m+1}(z)| < \varepsilon$.

Let $\hat{N} = N+1$. Then if $n \geq \hat{N}$ then $n \geq N+1$ and so $n-1 \geq N$ and so $|g_{n-1+1}(z)| < \varepsilon$ if $z \in A$; that is $|g_n(z)| < \varepsilon$, if $z \in A$.

In summary, if $n \geq \hat{N}$ and $z \in A$ then $|g_n(z) - \underbrace{f_0(z)}_{0}| < \varepsilon$

So, (g_n) converges to f_0 uniformly. $\blacksquare$

(Method 2 is on the next page)

$\downarrow$

⑥ (Method 2)

Suppose that $\sum_{k=1}^{\infty} g_k(z)$ converges uniformly on $A \subseteq \mathbb{C}$. Prove that the sequence $(g_k)_{k=1}^{\infty}$ converges uniformly to the zero function f_0 on A.

$[f_0 : A \to \mathbb{C},\ f(z) = 0\ \forall z \in A]$

<u>proof:</u> Let $\varepsilon > 0$. Let $S(z) = \sum_{k=1}^{\infty} g_k(z)$.

Let $S_n(z) = \sum_{k=1}^{n} g_k(z)$ be the n-th partial sum of $\sum_{k=1}^{\infty} g_k(z)$.

Since $\sum_{k=1}^{\infty} g_k(z)$ converges uniformly on A, there exists $N > 0$ where if $n \geq N$ then $|S_n(z) - S(z)| < \varepsilon/2$ for all $z \in A$.

Thus, if $n \geq N+1$ and $z \in A$, then

$$|g_n(z) - \underbrace{0}_{f_0(z)}| = |g_n(z)|$$

$$= \left| \sum_{k=1}^{n} g_k(z) - \sum_{k=1}^{n-1} g_k(z) \right|$$

$$= |S_n(z) - S_{n-1}(z)|$$

$$= |S_n(z) - S(z) + S(z) - S_{n-1}(z)|$$

$$\leq |S_n(z) - S(z)| + |S(z) - S_{n-1}(z)|$$

$$= |S_n(z) - S(z)| + |S_{n-1}(z) - S(z)|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

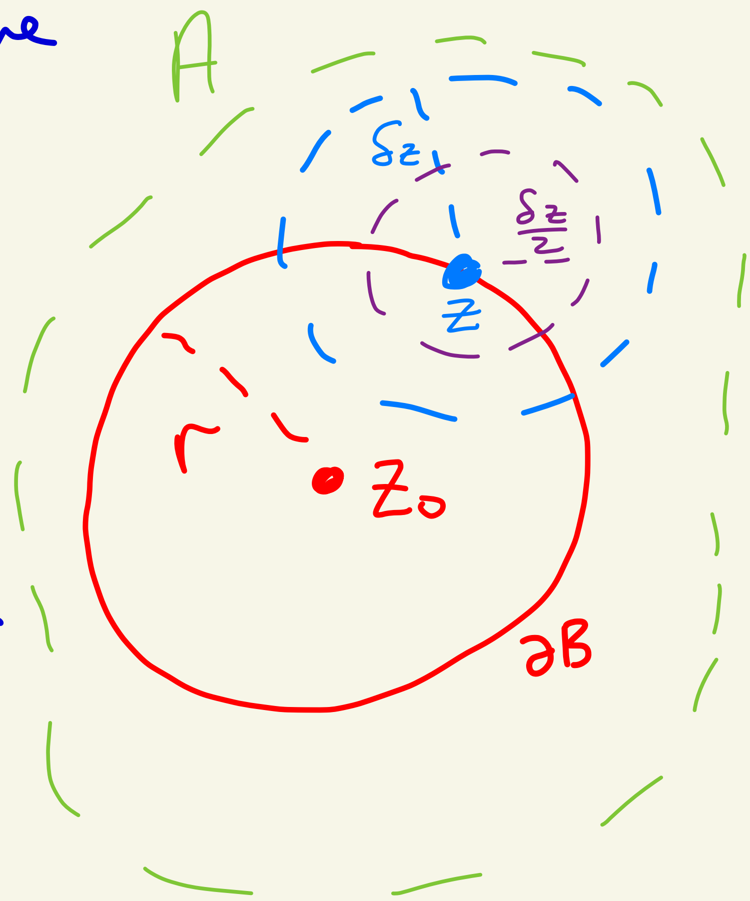
$$= \varepsilon.$$

$$\boxed{\begin{array}{l} n \geq N+1 \Rightarrow \\ \hline n-1 \geq N \\ n \geq N \end{array}}$$

Thus, $(g_n)_{n=1}^{\infty}$ converges uniformly to the zero function on A.

(A) Let ∂B denote the boundary of B, ie

$\partial B = \{ z \mid |z - z_0| = r \}$.



Since $B \subseteq A$ and A is open, for each $z \in \partial B$ there exists $\delta_z > 0$ where

$D(z; \delta_z) \subseteq A$

[Recall $D(z; \delta_z) = \{ w \mid |w - z| < \delta_z \}$].

Now we shrink that disc in half and look at $D(z; \frac{\delta_z}{2})$.

Consider the open cover

$$\mathcal{O} = \left\{ D\left(z; \frac{\delta_z}{2}\right) \,\middle|\, z \in \partial B \right\}$$

of ∂B.

Since ∂B is compact, there exists a finite subcover

$$\mathcal{O}' = \left\{ D\left(z_i ; \frac{\delta_{z_i}}{2}\right) \;\middle|\; i = 1, 2, \ldots, n \right\}$$

of ∂B.

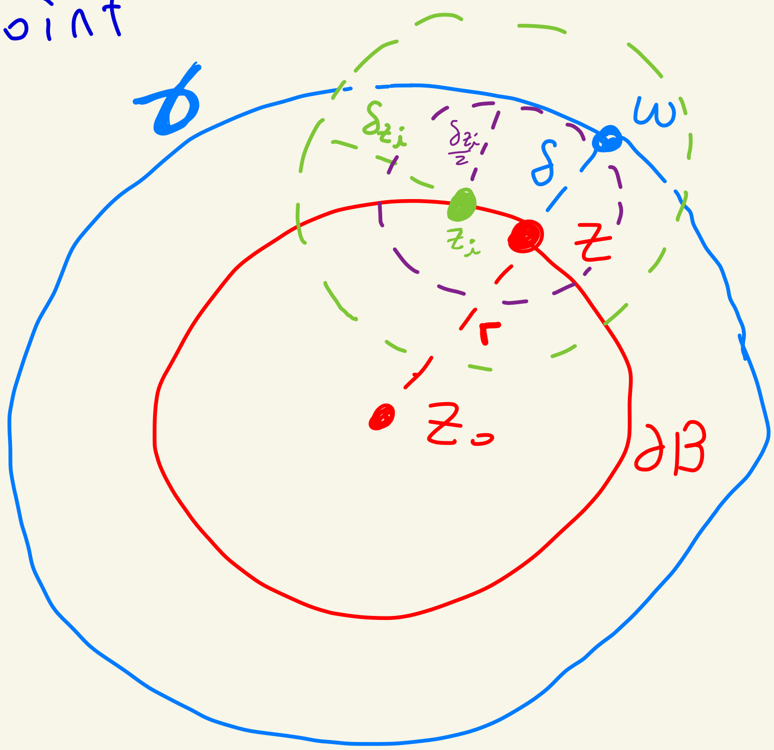
Let $\delta = \min\left\{ \frac{\delta_{z_i}}{2} \;\middle|\; i = 1, 2, \ldots, n \right\} > 0$

Let γ be the circle of radius $\rho = r + \delta$ centered at z_0

Since $\rho > r$, we have that γ contains B.

We now just have to show that γ is contained in A.

Let w be a point on γ. We must show $w \in A$.



Draw the line connecting w to z_0. This line intersects ∂B at some point z that satisfies $|w-z| = \delta$.

Since $z \in \partial B$, we know $z \in D(z_i; \frac{\delta_{z_i}}{2})$ for some i.

So, $|z - z_i| < \frac{\delta_{z_i}}{2}$. Then,

δ is min def

$$
\begin{aligned}
|w - z_i| &= |w - z + z - z_i| \\
&\le |w - z| + |z - z_i| \\
&< \frac{\delta_{z_i}}{2} + \delta \le \frac{\delta_{z_i}}{2} + \frac{\delta_{z_i}}{2} \\
&= \delta_{z_i}
\end{aligned}
$$

So, $w \in D(z_i; \delta_{z_i}) \subseteq A$.

Thus, all of γ is contained in A. $\blacksquare$