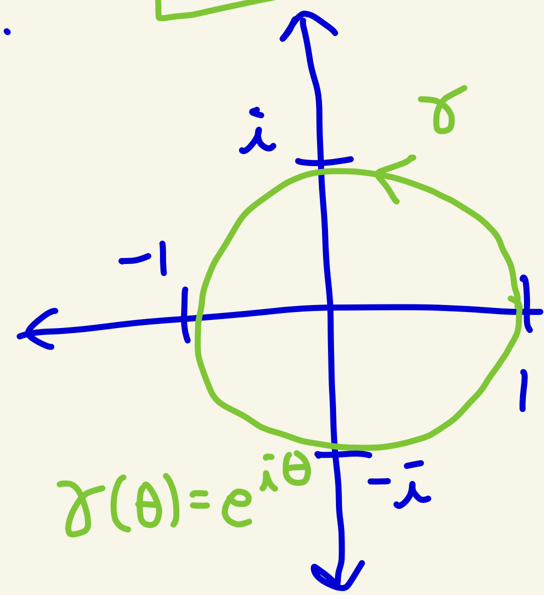
Math 5680
HW 6 Solutions

① $$\int_0^{2\pi} \frac{d\theta}{2-\sin(\theta)}$$

γ is the unit circle

Let $z = e^{i\theta}$ where $0 \leq \theta \leq 2\pi$.
Let γ be the curve traced out by this equation.



Then,

$$dz = ie^{i\theta}\, d\theta = iz\, d\theta$$

$$d\theta = \frac{dz}{iz}$$

$$\sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{z - \frac{1}{z}}{2i}$$

So,

$$\int_0^{2\pi} \frac{d\theta}{2-\sin(\theta)} = \int_\gamma \frac{\left(\frac{1}{iz}\right)}{2-\left(\frac{z-z^{-1}}{2i}\right)}\, dz =$$

$$= \int_\gamma \left(\frac{1}{iz}\right)\left(\frac{1}{2 - \frac{1}{2i}z + \frac{1}{2i}z^{-1}}\right) dz$$

$$= \int_\gamma \frac{dz}{2iz - \frac{1}{2}z^2 + \frac{1}{2}}$$

$$= \int_\gamma \frac{dz}{-\frac{1}{2}z^2 + 2iz + \frac{1}{2}}$$

$$= \int_\gamma \left(\frac{1}{-\frac{1}{2}}\right) \frac{dz}{z^2 - 4iz - 1}$$

$$= \int_\gamma \frac{-2\,dz}{(z - (2+\sqrt{3})i)(z - (2-\sqrt{3})i)}$$

If the roots of $az^2 + bz + c = 0$ are z_1, z_2 then

$$az^2 + bz + c = a(z - z_1)(z - z_2)$$

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{4i \pm \sqrt{(-4i)^2 - 4(1)(-1)}}{2 \cdot 1}$$

$$= \left(4i \pm \sqrt{-16 + 4}\right)/2$$

$$= \left(4i \pm \sqrt{-12}\right)/2$$

$$= \left(4i \pm \sqrt{12}\, i\right)/2$$

$$= \left(4i \pm 2\sqrt{3}\, i\right)/2$$

$$= (2 + \sqrt{3})i,$$

$$(2 - \sqrt{3})i$$

$$(2 + \sqrt{3})i \approx 3{,}73i$$

$$(2 - \sqrt{3})i \approx 0{,}27i$$

$$= -2 \int_\gamma \frac{dz}{(z-z_1)(z-z_2)}$$

where $z_1 = (2-\sqrt{3})i$
$z_2 = (2+\sqrt{3})i$

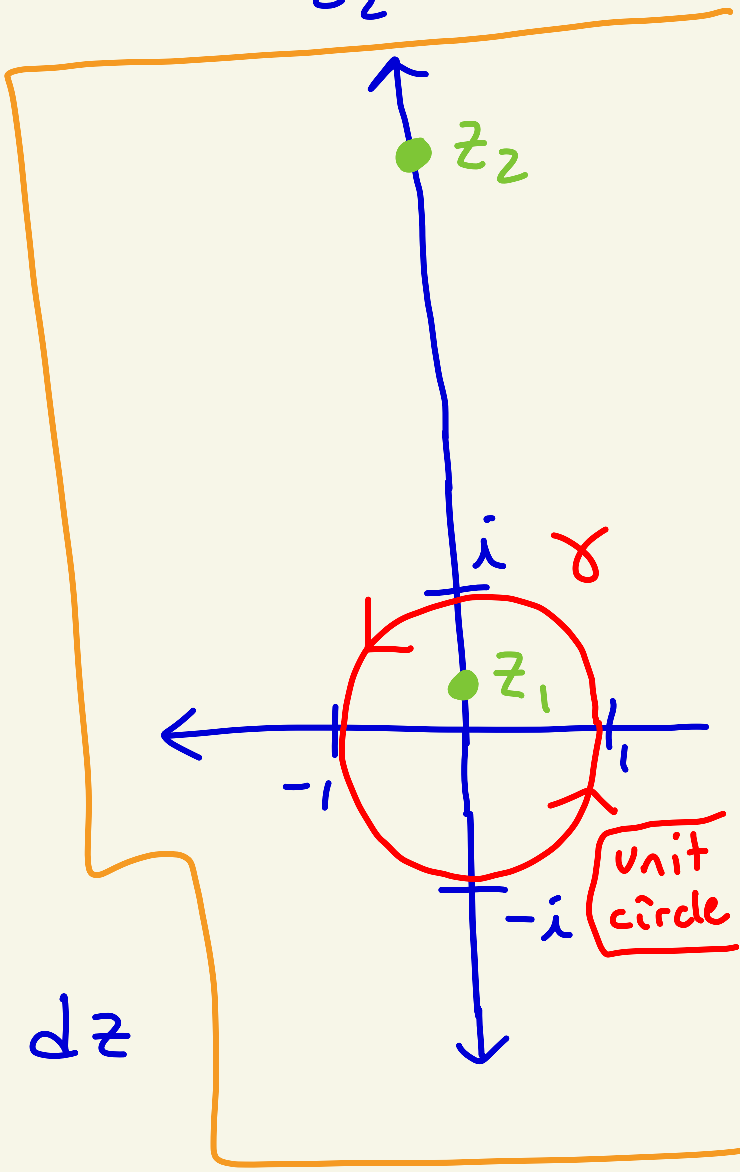
$z_1 \approx 0.27i$

$z_2 \approx 3.73i$

Note that z_1 is inside γ but z_2 is outside γ.

So, by the residue theorem we have that

$$\int_0^{2\pi} \frac{d\theta}{2-\sin(\theta)}$$



$$= -2 \int_\gamma \frac{1}{(z-z_1)(z-z_2)} \, dz$$

$$= -2 \cdot 2\pi i \cdot \text{Res}\left[\frac{1}{(z-z_1)(z-z_2)} ; z_1\right]$$

$$= -4\pi i \cdot \operatorname{Res}\left[\frac{\overbrace{1/(z-z_2)}^{\varphi(z)}}{(z-z_1)} ; z_1 \right]$$

Let $\varphi(z) = \dfrac{1}{z-z_2}$.

$\left[\dfrac{\varphi(z)}{z-z_1} \right.$ has a simple pole at z_1 since φ is analytic at z_1 and $\left. \varphi(z_1) = \dfrac{1}{z_1 - z_2} \neq 0. \right]$

Then, $\displaystyle\int_0^{2\pi} \frac{d\theta}{2-\sin(\theta)}$ becomes

$\varphi(z) = \dfrac{1}{z-z_2}$

$$-4\pi i \operatorname{Res}\left[\frac{\boxed{\varphi(z)}}{z-z_1} ; z_1 \right] = -4\pi i\, \varphi(z_1)$$

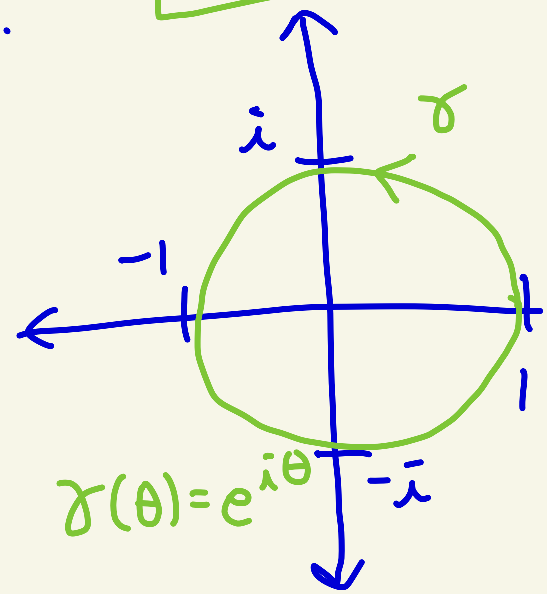
$$= -4\pi i \frac{1}{z_1 - z_2} = \frac{-4\pi i}{(2-\sqrt{3})i - (2+\sqrt{3})i}$$

$$= \frac{-4\pi i}{-2\sqrt{3}i} = \frac{2\pi}{\sqrt{3}} = \boxed{\frac{2\sqrt{3}\,\pi}{3}}$$

② $\displaystyle\int_0^{2\pi} \frac{\cos(3\theta)}{5-4\cos(\theta)}\,d\theta$

γ is the unit circle

Let $z = e^{i\theta}$ where $0 \leq \theta \leq 2\pi$.
Let γ be the curve traced out by this equation.



Then,

$$dz = ie^{i\theta}\,d\theta = iz\,d\theta$$

$$d\theta = \frac{dz}{iz}$$

$$\cos(\theta) = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+\frac{1}{z}}{2} = \frac{z+z^{-1}}{2}$$

$$\cos(3\theta) = \frac{e^{i3\theta}+e^{-i3\theta}}{2} = \frac{z^3+z^{-3}}{2}$$

So,

$$\int_0^{2\pi} \frac{\cos(3\theta)}{5-4\cos(\theta)}\,d\theta = \int_\gamma \frac{\left(\frac{z^3+z^{-3}}{2}\right)}{5-4\left(\frac{z+z^{-1}}{2}\right)}\,\frac{dz}{iz}$$

$$= \frac{1}{i} \int_{\gamma} \frac{\frac{1}{2}\left(z^{3} + z^{-3}\right)}{5 - 2z - 2z^{-1}} \, \frac{dz}{z}$$

$$= \frac{1}{2i} \int_{\gamma} \frac{z^{3} + z^{-3}}{5z - 2z^{2} - 2} \, dz \qquad \times \frac{z^{3}}{z^{3}}$$

$$= \frac{1}{2i} \int_{\gamma} \frac{z^{6} + 1}{-2z^{5} + 5z^{4} - 2z^{3}} \, dz$$

$$= -\frac{1}{2i} \int_{\gamma} \frac{z^{6} + 1}{2z^{5} - 5z^{4} + 2z^{3}} \, dz$$

$$= -\frac{1}{2i} \int_{\gamma} \frac{z^{6} + 1}{z^{3}\left(2z^{2} - 5z + 2\right)} \, dz$$

$$= -\frac{1}{2i} \int_{\gamma} \frac{z^{6} + 1}{z^{3}(2z-1)(z-2)} \, dz$$

The function $f(z) = \dfrac{z^6+1}{z^3(2z-1)(z-2)}$ has poles at $z = 0, \frac{1}{2}, 2$. And the poles at $0, \frac{1}{2}$ are inside γ.

Pole at 0:

$$f(z) = \frac{(z^6+1)/[(2z-1)(z-2)]}{z^3} = \frac{\varphi_1(z)}{z^3}$$

where φ_1 is analytic at 0 and

$$\varphi_1(0) = \frac{1}{-2} \neq 0.$$

Thus, f has a pole of order 3 at 0 and

$$\mathrm{Res}(f;0) = \frac{\varphi_1^{(3-1)}(0)}{(3-1)!} = \frac{\varphi_1^{(2)}(0)}{2}$$

Now,

$$\varphi_1(z) = (z^6+1)(2z-1)^{-1}(z-2)^{-1}$$

Using Mathematica I got that

$$\varphi_1''(z) = \frac{6(7-10z+4z^2+20z^4-80z^5+106z^6-50z^7+8z^8)}{(-2+z)^3(-1+2z)^3}$$

So,

$$\text{Res}(f;0) = \frac{\varphi_1''(0)}{2} = \frac{\left(\frac{6(7)}{(-2)^3(-1)^3}\right)}{2} = \frac{42}{16} = \frac{21}{8}$$

Pole at $\frac{1}{2}$:

$$f(z) = \frac{(z^6+1)\big/2z^3(z-2)}{(z-1/2)} = \frac{\varphi_2(z)}{2z-1}$$

where φ_2 is analytic at $1/2$ and

$$\varphi_2(1/2) = (1/2^6+1)\big/2\cdot\left(\tfrac{1}{2}\right)^3\left(\tfrac{1}{2}-2\right) = -\frac{65}{24} \neq 0$$

Thus, f has a simple pole at $\frac{1}{2}$ and

$$\mathrm{Res}\left(f; \frac{1}{2}\right) = \frac{\varphi_2^{(1-1)}\left(\frac{1}{2}\right)}{(1-1)!} = \varphi_2\left(\frac{1}{2}\right) = \frac{-65}{24}$$

Thus,

$$\int_0^{2\pi} \frac{\cos(3\theta)}{5-4\cos(\theta)}\, d\theta = -\frac{1}{2i} \int_\gamma \frac{z^6+1}{z^3(2z-1)(z-2)}\, dz$$

$$= -\frac{1}{2i}\, 2\pi i \left[\mathrm{Res}(f;0) + \mathrm{Res}\left(f; \frac{1}{2}\right) \right]$$

$$= -\pi \left[\frac{21}{8} - \frac{65}{24} \right] = \boxed{\frac{\pi}{12}}$$

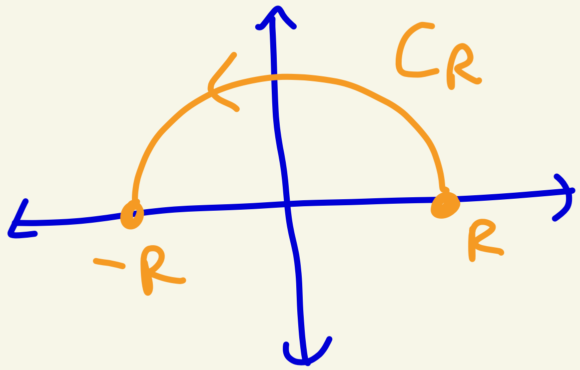
③ $\displaystyle\int_0^\infty \frac{dx}{1+x^6}$

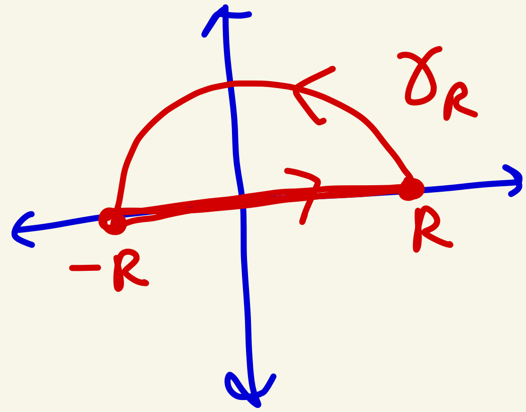
Note that $f(x) = \dfrac{1}{1+x^6}$ is an even function since $f(-x) = f(x)$.

So, $\displaystyle 2\int_0^\infty \frac{dx}{1+x^6} = \lim_{R\to\infty} \int_{-R}^{R} \frac{dx}{1+x^6}$

Let $R>1$, and C_R be the top half of the circle $|z|=R$ oriented counterclockwise.

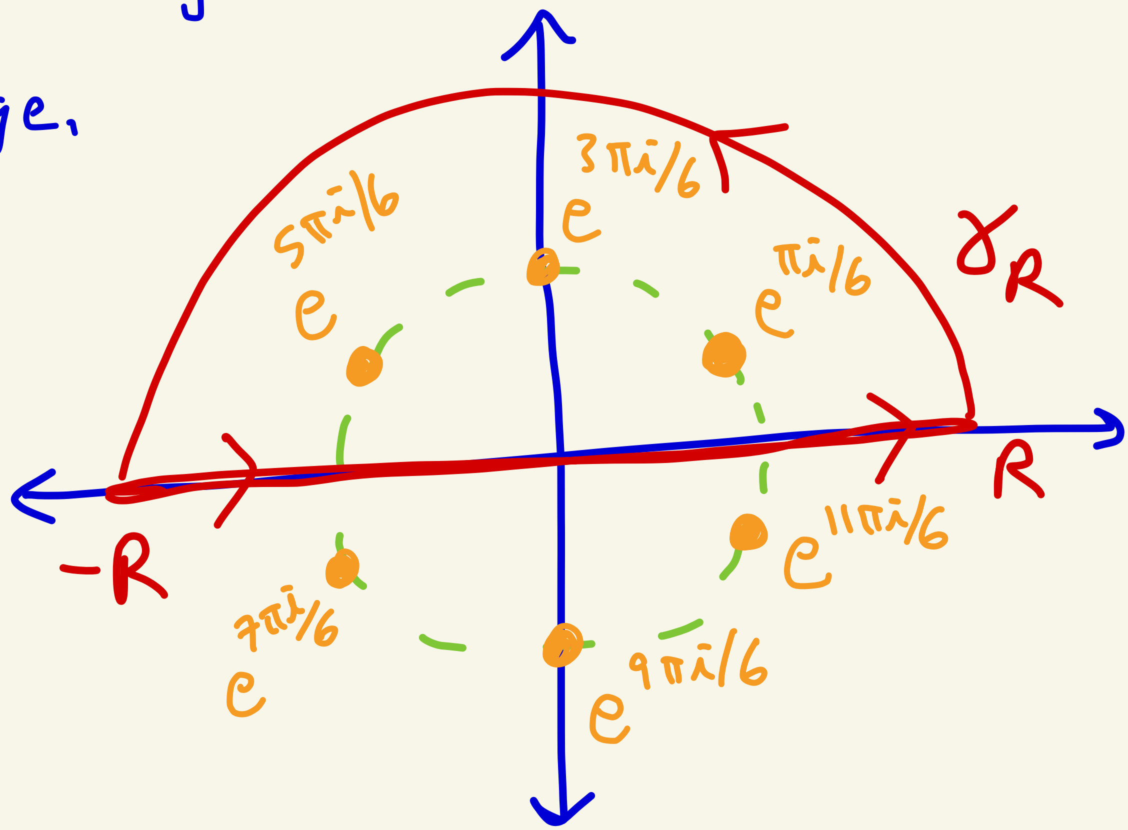
Let γ_R be C_R followed by the line connecting $-R$ to R.





The poles of $\frac{1}{1+z^6}$ are pictured in in this diagram $\rightarrow$ in orange.

Each pole is a simple pole.



The poles inside the curve γ_R are $e^{\pi i/6}, e^{3\pi i/6}, e^{5\pi i/6}$.

Recall this theorem!

Let g, h be analytic at z_0 with $g(z_0) \neq 0$, $h(z_0) = 0$, $h'(z_0) \neq 0$. Then $f(z) = g(z)/h(z)$ has a simple pole at z_0 and

$$\mathrm{Res}(f; z_0) = \frac{g(z_0)}{h'(z_0)}$$

So,

$$\operatorname{Res}\left(\frac{1}{1+z^6}; e^{\pi i/6}\right) = \frac{1}{6\left(e^{\pi i/6}\right)^5} = \frac{1}{6}e^{-5\pi i/6}$$

$$= \frac{1}{6}\left[\cos\left(-\frac{5\pi}{6}\right) + i\sin\left(-\frac{5\pi}{6}\right)\right] = \frac{1}{6}\left[-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right]$$

$$= -\frac{\sqrt{3}}{12} - \frac{1}{12}i$$

$$\operatorname{Res}\left(\frac{1}{1+z^6}; e^{3\pi i/6}\right) = \frac{1}{6\left(e^{3\pi i/6}\right)^5} = \frac{1}{6}e^{-5\pi i/2}$$

$$= \frac{1}{6}\left[\underbrace{\cos\left(-\frac{5\pi}{2}\right)}_{0} + i\underbrace{\sin\left(-\frac{5\pi}{2}\right)}_{-1}\right] = -\frac{1}{6}i$$

$$\operatorname{Res}\left(\frac{1}{1+z^6}; e^{5\pi i/6}\right) = \frac{1}{6\left(e^{5\pi i/6}\right)^5} = \frac{1}{6}e^{-25\pi i/6}$$

$$= \frac{1}{6}\left[\cos\left(-\frac{25\pi}{6}\right) + i\sin\left(-\frac{25\pi}{6}\right)\right] = \frac{1}{6}\left[\frac{\sqrt{3}}{2} - \frac{1}{2}i\right]$$

$$= \frac{\sqrt{3}}{12} - \frac{1}{12}i$$

Thus,

$$\int_{\gamma_R} \frac{dz}{1+z^6} = 2\pi i\left[-\frac{\sqrt{3}}{12} - \frac{i}{12} - \frac{1}{6}i + \frac{\sqrt{3}}{12} - \frac{i}{12}\right]$$

$$= 2\pi i\left[-\frac{1}{3}i\right] = \frac{2\pi}{3}$$

Since $\displaystyle\int_{\gamma_R} \frac{dz}{1+z^6} = \int_{-R}^{R} \frac{dz}{1+z^6} + \int_{C_R} \frac{dz}{1+z^6}$

we have

$$\boxed{\int_{-R}^{R} \frac{dz}{1+z^6} + \int_{C_R} \frac{dz}{1+z^6} = \frac{2\pi}{3}} \quad (*)$$

Now we want to let $R \to \infty$.

Note that if z is on C_R then $|z| = R$ and so

$$\left|\frac{1}{1+z^6}\right| = \frac{1}{|z^6+1|} \leq \frac{1}{\left||z|^6-1\right|}$$

$|a+b| \geq \left||a|-|b|\right|$

$$= \frac{1}{R^6-1}$$

$R>1$, so
$\left||z|^6-1\right| = |R^6-1| = R^6-1$

So, $$\left|\int_{C_R} \frac{dz}{1+z^6}\right| \leq \frac{1}{R^6-1} \cdot \text{length}(C_R)$$

$$= \frac{\pi R}{R^6-1} \longrightarrow 0$$

as $R \to \infty$

So, taking $R \to \infty$ in (*) gives

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^6} = \frac{2\pi}{3}$$

Thus,

$$\int_{0}^{\infty} \frac{dx}{1+x^6} = \frac{1}{2} \int_{-\infty}^{\infty} \frac{dx}{1+x^6}$$

$$= \frac{1}{2}\left(\frac{2\pi}{3}\right) = \frac{\pi}{3}$$

④ $\int_0^\infty \frac{1+x^2}{1+x^4}\,dx$

Note that $f(x) = \frac{1+x^2}{1+x^4}$ is an even function since $f(-x) = f(x)$.

So, $2\int_0^\infty \frac{1+x^2}{1+x^4} = \lim_{R\to\infty} \int_{-R}^{R} \frac{1+x^2}{1+x^4}$

Let $R>1$, and C_R be the top half of the circle $|z|=R$ oriented counterclockwise.

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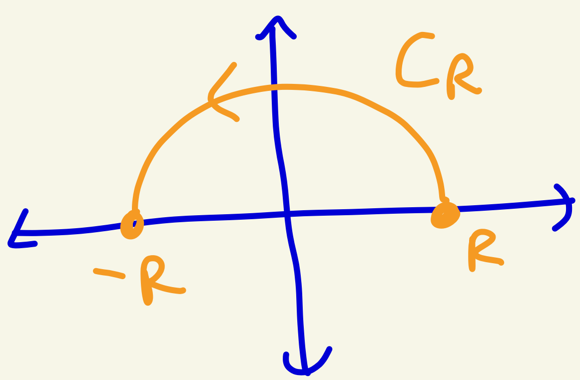
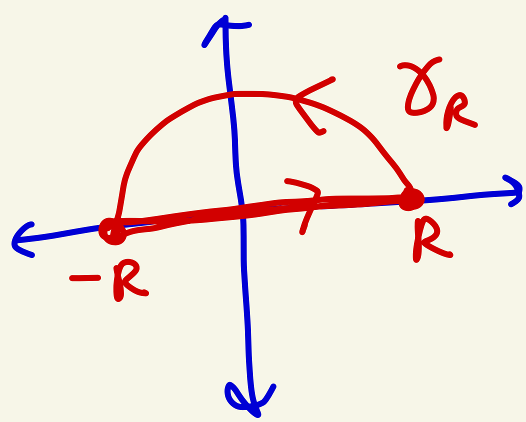




The poles of $\dfrac{1+z^2}{1+z^4}$ are when

$$z^4 = -1 = 1 \cdot e^{\pi i}$$

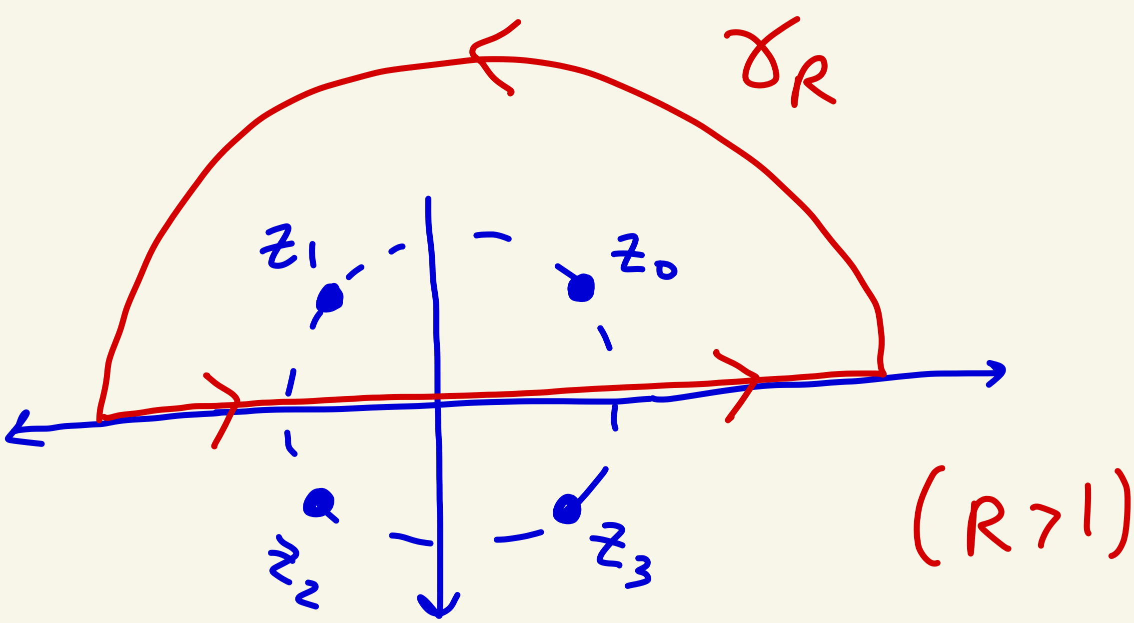
$$z_k = 1^{1/4} \cdot e^{\left(\frac{\pi}{4} + \frac{2\pi k}{4}\right)i}, \quad k = 0, 1, 2, 3$$

$$z_0 = e^{\frac{\pi}{4}i} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$$

$$z_1 = e^{\frac{3\pi}{4}i} = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$$

$$z_2 = e^{\frac{5\pi}{4}i} = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$$

$$z_3 = e^{\frac{7\pi}{4}i} = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$$



z_0, z_1 are the poles lying inside of γ_R

Thus,

$$\int_{\gamma_R} \frac{1+x^2}{1+x^4}\,dx = 2\pi i \operatorname{Res}(f; z_0) + 2\pi i \operatorname{Res}(f; z_1)$$

Note that $f(z) = \dfrac{g(z)}{h(z)}$ where

$g(z) = 1+z^2$, $h(z) = 1+z^4$

Since $g(z_0) = 1+\left(e^{\frac{\pi}{4}i}\right)^2 = 1+e^{\frac{\pi}{2}i} = 1+i \neq 0$

$h(z_0) = 0$

$h'(z_0) = 4\left(e^{\frac{\pi}{4}i}\right)^3 = 4e^{\frac{3\pi}{4}i} \neq 0$

we see that z_0 is a simple pole and

$$\operatorname{Res}(f; z_0) = \frac{g(z_0)}{h'(z_0)} = \frac{1+e^{\frac{\pi}{2}i}}{4e^{3\pi/4\, i}}$$

$$= \frac{1}{4}\left[e^{-\frac{3\pi}{4}i} + e^{-\frac{\pi}{4}i}\right]$$

$$= \frac{1}{4}\left[-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i + \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right]$$

$$= -\frac{\sqrt{2}}{4}i$$

Since $g(z_1) = 1 + \left(e^{\frac{3\pi}{4}i}\right)^2 = 1 + e^{\frac{3\pi}{2}i} = 1 - i \neq 0$

$h(z_1) = 0$

$h'(z_1) = 4\left(e^{\frac{3\pi}{4}i}\right)^3 = 4e^{\frac{9\pi}{4}i} = 4e^{\frac{\pi}{4}i} \neq 0$

we see that z_1 is a simple pole and

$$\operatorname{Res}(f; z_0) = \frac{g(z_1)}{h'(z_1)} = \frac{1 + e^{\frac{3\pi}{2}i}}{4e^{\frac{\pi}{4}i}}$$

$$= \frac{1}{4}\left[e^{-\frac{\pi}{4}i} + e^{\frac{5\pi}{4}i}\right]$$

$$= \frac{1}{4}\left[\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i - \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right]$$

$$= \frac{1}{4}\left[-\sqrt{2}i\right] = \frac{-\sqrt{2}}{4}i$$

So,

$$\int_{\gamma_R} \frac{1+z^2}{1+z^4}\,dz = 2\pi i\left(-\frac{\sqrt{2}}{4}i\right) + 2\pi i\left(-\frac{\sqrt{2}}{4}i\right)$$

$$= \frac{\sqrt{2}}{2}\pi + \frac{\sqrt{2}}{2}\pi = \sqrt{2}\pi$$

Since

$$\int_{\gamma_R} \frac{1+z^2}{1+z^4}\,dz = \int_{-R}^{R} \frac{1+z^2}{1+z^4}\,dz + \int_{C_R} \frac{1+z^2}{1+z^4}\,dz$$

we have

$$\boxed{\int_{-R}^{R} \frac{1+z^2}{1+z^4}\,dz + \int_{C_R} \frac{1+z^2}{1+z^4}\,dz = \sqrt{2}\pi} \quad (*)$$

Now we want to let $R \to \infty$.

Note that if $z \in C_R$ then $|z| = R$

and $|1+z^2| \leq |1| + |z^2| = 1 + |z|^2 = 1 + R^2$

and $|1+z^4| \geq \big|\,|1| - |z^4|\,\big| = \big|1 - |z|^4\big|$

$= |1 - R^4| = |R^4 - 1| = R^4 - 1$

since $R > 1$

So,

$$\left|\int_{C_R} \frac{1+z^2}{1+z^4}\,dz\right| \leq \frac{1+R^2}{R^4-1} \cdot \text{arclength}(C_R)$$

$$= \frac{1+R^2}{R^4-1} \cdot \pi R = \pi \left[\frac{R+R^2}{R^4-1}\right]$$

$$= \pi \left[\frac{\frac{1}{R^3}+\frac{1}{R^2}}{1-\frac{1}{R^4}}\right] \longrightarrow \pi\left(\frac{0+0}{1-0}\right) = 0$$

as $R \to \infty$.

So, (*) becomes

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{1+z^2}{1+z^4}\,dz + 0 = \sqrt{2}\,\pi$$

Thus,

$$\int_0^\infty \frac{1+x^2}{1+x^4}\,dx = \frac{1}{2}\lim_{R\to\infty}\int_{-R}^{R}\frac{1+x^2}{1+x^4}\,dx$$

$$= \frac{1}{2}\sqrt{2}\,\pi = \frac{\sqrt{2}}{2}\pi$$

5 Let $g(x) = \dfrac{x\sin(x)}{x^4+1}$

Since $g(-x) = \dfrac{(-x)\sin(-x)}{(-x)^4+1} = \dfrac{(-x)(-\sin(x))}{x^4+1}$

$$= \frac{x\sin(x)}{x^4+1} = g(x)$$

we see that g is even.

Hence if $\displaystyle\lim_{R\to\infty}\int_{-R}^{R} \frac{x\sin(x)}{x^4+1}\,dx$

exists then $\displaystyle\lim_{R\to\infty}\int_{-R}^{R} \frac{x\sin(x)}{x^4+1}\,dx = \int_{-\infty}^{\infty} \frac{x\sin(x)}{x^4+1}\,dx$

from in class discussion.

Let $f(z) = \dfrac{z}{z^4+1}$. Then if $x \in \mathbb{R}$

we have

$$\operatorname{Im}\left[f(x)e^{ix}\right] = \operatorname{Im}\left[\frac{x}{x^4+1}\left(\cos(x) + i\sin(x)\right)\right]$$

$$= \frac{x}{x^4+1}\sin(x)$$

Note that $f(z)e^{iz} = \frac{z}{z^4+1}e^{iz}$ is analytic on $\mathbb{C} - \{e^{\frac{\pi}{4}i}, e^{\frac{3\pi}{4}i}, e^{\frac{5\pi}{4}i}, e^{\frac{7\pi}{4}i}\}$

$z^4+1=0$ when $z = e^{\frac{\pi}{4}i}, e^{\frac{3\pi}{4}i}, e^{\frac{5\pi}{4}i}, e^{\frac{7\pi}{4}i}$

see prob. #4 for how to get these roots

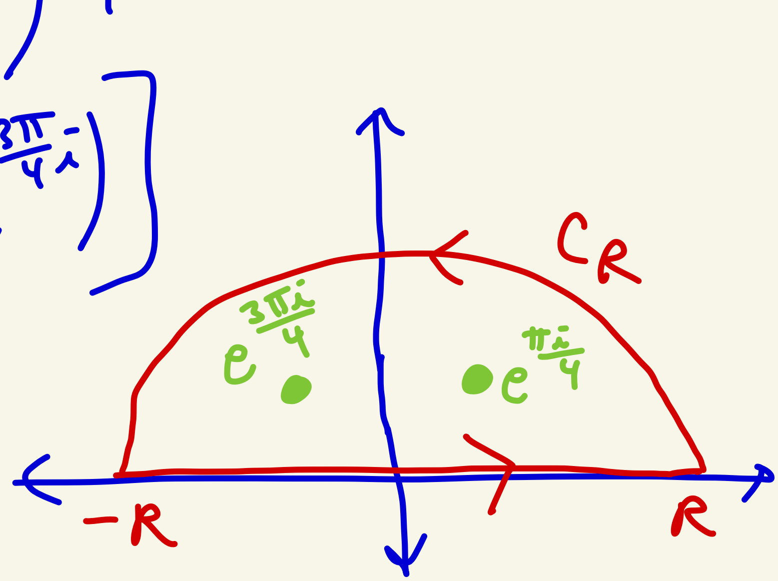
Let C_R denote the upper-half of the circle $|z|=R$ oriented counterclockwise.

Then,

$$\int_{C_R} \frac{z}{z^4+1}e^{iz}\,dz + \int_{-R}^{R} \frac{x}{x^4+1}e^{ix}\,dx$$

$f(x)$ we write x because this integral is on the real line and $z = x+i0$ there

$$= 2\pi i\left[\operatorname{Res}\left(\frac{z}{z^4+1}e^{iz}; e^{\frac{\pi i}{4}}\right) + \operatorname{Res}\left(\frac{z}{z^4+1}e^{iz}; e^{\frac{3\pi}{4}i}\right)\right]$$



Note that

$$f(z)e^{iz} = \frac{ze^{iz}}{z^4+1} = \frac{g(z)}{h(z)}$$

where $g(e^{\pi i/4}) = e^{\pi i/4} e^{ie^{\pi i/4}} = e^{\frac{\pi i}{4}} e^{i(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2})} \neq 0$

and $g(e^{3\pi i/4}) = e^{3\pi i/4} e^{ie^{3\pi i/4}} = e^{3\pi i/4} e^{i(-\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2})} \neq 0$

$h(e^{\pi i/4}) = h(e^{3\pi i/4}) = 0$

$h'(e^{\pi i/4}) = 4(e^{\pi i/4})^3 = 4e^{3\pi i/4} \neq 0$

$h'(e^{3\pi i/4}) = 4(e^{3\pi i/4})^3 = 4e^{9\pi i/4} = 4e^{\pi i/4} \neq 0$

Thus, $e^{\pi i/4}$ and $e^{3\pi i/4}$ are simple poles

and

$$\operatorname{Res}\left(\frac{ze^{iz}}{z^4+1}; e^{\pi i/4}\right) = \frac{g(e^{\pi i/4})}{h'(e^{\pi i/4})}$$

$$= \frac{e^{\pi i/4} e^{ie^{\pi i/4}}}{4e^{3\pi i/4}} = \frac{1}{4} \underbrace{e^{\frac{-2\pi i}{4}}}_{e^{-\frac{\pi}{2}i} = -i} e^{i(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2})}$$

$$= \frac{1}{4}(-i)e^{i\frac{\sqrt{2}}{2}} e^{-\frac{\sqrt{2}}{2}} =$$

$$= -\frac{1}{4} i e^{-\frac{\sqrt{2}}{2}} \left[\cos\left(\frac{\sqrt{2}}{2}\right) + i\sin\left(\frac{\sqrt{2}}{2}\right)\right]$$

$$= -\frac{1}{4} i e^{-\frac{\sqrt{2}}{2}} \cos\left(\frac{\sqrt{2}}{2}\right) + \frac{1}{4} e^{-\frac{\sqrt{2}}{2}} \sin\left(\frac{\sqrt{2}}{2}\right)$$

And,

$$\operatorname{Res}\left(\frac{ze^{iz}}{z^4+1}; e^{\frac{3\pi i}{4}}\right) = \frac{g(e^{\frac{3\pi i}{4}})}{h'(e^{3\pi i/4})}$$

$$= \frac{e^{3\pi i/4} e^{i e^{3\pi i/4}}}{4e^{\pi i/4}} = \frac{1}{4} \underbrace{e^{\frac{2\pi i}{4}}}_{e^{\frac{\pi}{2} i} = i} e^{i\left(-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)}$$

$$= \frac{1}{4}(i) e^{-i\frac{\sqrt{2}}{2}} e^{-\frac{\sqrt{2}}{2}}$$

$$= \frac{1}{4} i e^{-\frac{\sqrt{2}}{2}} \left[\underbrace{\cos\left(-\frac{\sqrt{2}}{2}\right)}_{\cos\left(\frac{\sqrt{2}}{2}\right)} + \underbrace{i\sin\left(-\frac{\sqrt{2}}{2}\right)}_{-\sin\left(\frac{\sqrt{2}}{2}\right)}\right]$$

$$= \frac{1}{4} i e^{-\frac{\sqrt{2}}{2}} \cos\left(\frac{\sqrt{2}}{2}\right) + \frac{1}{4} e^{-\frac{\sqrt{2}}{2}} \sin\left(\frac{\sqrt{2}}{2}\right)$$

Therefore

$$\int_{C_R} f(z)e^{iz}dz + \int_{-R}^{R} \frac{x}{x^2+1} e^{ix}dx$$

$$= 2\pi i\left[-\frac{1}{4}ie^{-\frac{\sqrt{2}}{2}}\cos\left(\frac{\sqrt{2}}{2}\right) + \frac{1}{4}e^{-\frac{\sqrt{2}}{2}}\sin\left(\frac{\sqrt{2}}{2}\right) + \frac{1}{4}ie^{-\frac{\sqrt{2}}{2}}\cos\left(\frac{\sqrt{2}}{2}\right) + \frac{1}{4}e^{-\frac{\sqrt{2}}{2}}\sin\left(\frac{\sqrt{2}}{2}\right)\right]$$

$$= 2\pi i\left[\frac{1}{2}e^{-\frac{\sqrt{2}}{2}}\sin\left(\frac{\sqrt{2}}{2}\right)\right]$$

$$= i\pi e^{-\frac{\sqrt{2}}{2}}\sin\left(\frac{\sqrt{2}}{2}\right)$$

Taking imaginary parts of both sides of the previous page we get

$$\operatorname{Im}\left(\int_{C_R} f(z)e^{iz}\,dz\right) + \int_{-R}^{R} \frac{x}{x^4+1}\sin(x)\,dx = \pi e^{-\frac{\sqrt{2}}{2}}\sin\left(\frac{\sqrt{2}}{2}\right)$$

Suppose $z = x+iy$ is on C_R with $R>1$.

Then $|z| = R$ and

$$|z^4+1| \geq \big||z^4|-|1|\big| = \big||z|^4-1\big| = |R^4-1| \underbrace{=}_{\text{since } R>1} R^4-1$$

So, for such a z and R we have

$$|f(z)e^{iz}| = \left|\frac{z}{z^4+1}\right|\left|e^{ix-y}\right| = \frac{|z|}{|z^4+1|}\cdot \underbrace{|e^{ix}|}_{1}\underbrace{|e^{-y}|}_{e^{-y}}$$

$$\leq \frac{R}{R^4-1}\cdot 1 = \frac{R}{R^4-1}$$

$y \geq 0$ since z is on C_R in the upper half plane
So $e^{-y} \leq 1$

So,

$$\left|\int_{C_R} f(z)e^{iz}\,dz\right| \leq \frac{R}{R^4-1}\cdot \underbrace{\pi R}_{\text{arclength}(C_R)}$$

$$= \frac{\pi R^2}{R^4-1} = \frac{\pi \frac{1}{R^2}}{1-\frac{1}{R^4}} \to \frac{\pi\cdot 0}{1-0} = 0$$

as $R\to\infty$

Thus, $\displaystyle\lim_{R\to\infty}\int_{C_R} f(z)e^{iz}\,dz = 0$

So, $\displaystyle\lim_{R\to\infty} \operatorname{Im}\left(\int_{C_R} f(z)e^{iz}\,dz\right) = 0$

So,

$$\int_{-\infty}^{\infty}\frac{x\sin(x)}{x^4+1}\,dx = \lim_{R\to\infty}\int_{-R}^{R}\frac{x\sin(x)}{x^4+1}\,dx$$

$$= \boxed{\pi e^{-\sqrt{2}/2}\sin\left(\frac{\sqrt{2}}{2}\right)}$$