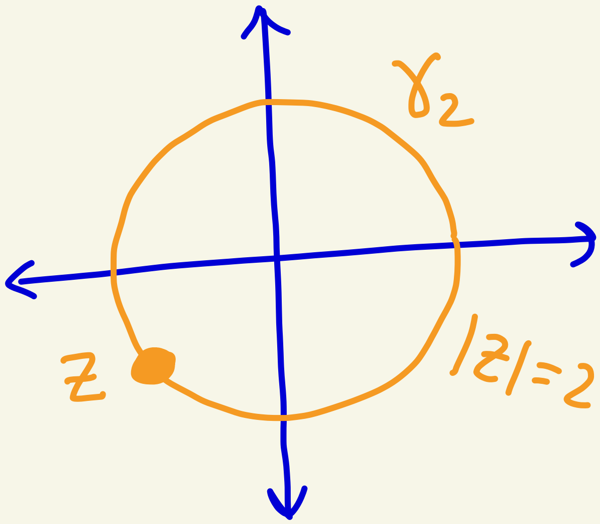
Math 5680
HW 8 Solutions

①

$\boxed{\text{Part (i)}}$

First we show that all the zeros of $p(z) = z^6 - 5z^2 + 10$ lie inside of the disc $|z| < 2$.

Let γ_2 be the circle $|z| = 2$



Let $f(z) = z^6$

and $h(z) = -5z^2 + 10$

If z is on γ_2, then $|z| = 2$ and so

$$|f(z)| = |z^6| = |z|^6 = 2^6 = 64$$

and

$$|h(z)| = |-5z^2 + 10| \leq |-5z^2| + |10|$$
$$= 5|z|^2 + 10 = 5 \cdot 2^2 + 10 = 30$$

Thus,

$$|h(z)| = 30 < 64 = |f(z)|$$

for all z on γ_2.

Thus, $f(z) = z^6$ and

$p(z) = f(z) + h(z) = z^6 - 5z^2 + 10$

have the same number of zeros (counting multiplicity) inside γ_2.

We know $f(z) = z^6$ has a zero at $z_0 = 0$ of multiplicity 6 and those are its only zeros inside γ_2.

Thus, $p(z)$ has 6 zeroes (counting multiplicity) inside of $|z| < 2$.

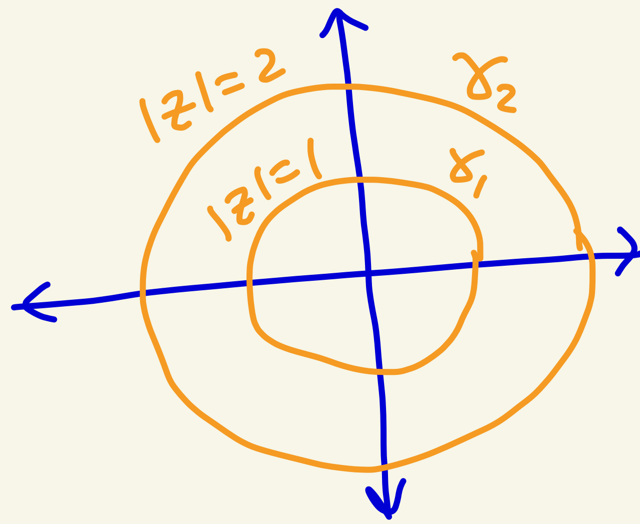
Since p is a degree 6 polynomial it can't have any more zeros.

Thus, all of the zeros of $p(z)$ are inside $|z|<2$

$\boxed{\text{Part (ii)}}$

Let $h(z) = -5z^2$ and $f(z) = z^6 + 10$.

Let γ_1 be the circle $|z|=1$.



If z is on γ_1,
then $|z|=1$ and so

$$|h(z)| = |-5z^2| = 5|z|^2 = 5 \cdot 1^2 = 5$$

and

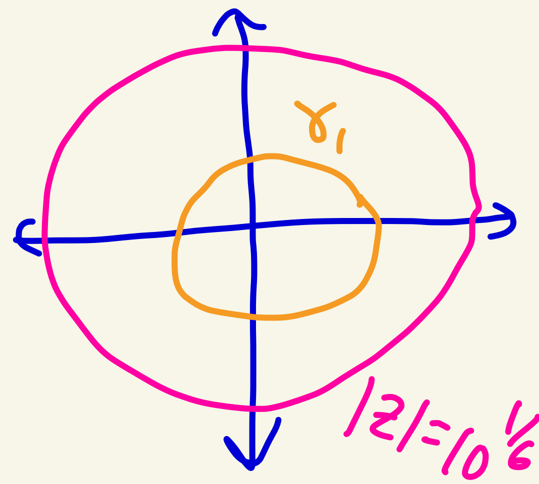
$$|f(z)| = |z^6+10| \geq \big||z^6|-|10|\big| = \big||z|^6-10\big| = |1^6-10| = 9$$

So, if z is on γ_1 then
$|h(z)| = 5 < 9 \leq |f(z)|$.

By Rouché's theorem, $f(z) = z^6 + 10$ and $p(z) = h(z) + f(z) = z^6 - 5z^2 + 10$ have the same number of zeros (counting multiplicity) inside γ_1.

How many zeros does f have inside γ_1?

Suppose $f(z) = 0$, ie $z^6 + 10 = 0$.



Then $z^6 = -10$.

So, $|z^6| = 10$.

Thus, $|z|^6 = 10$.

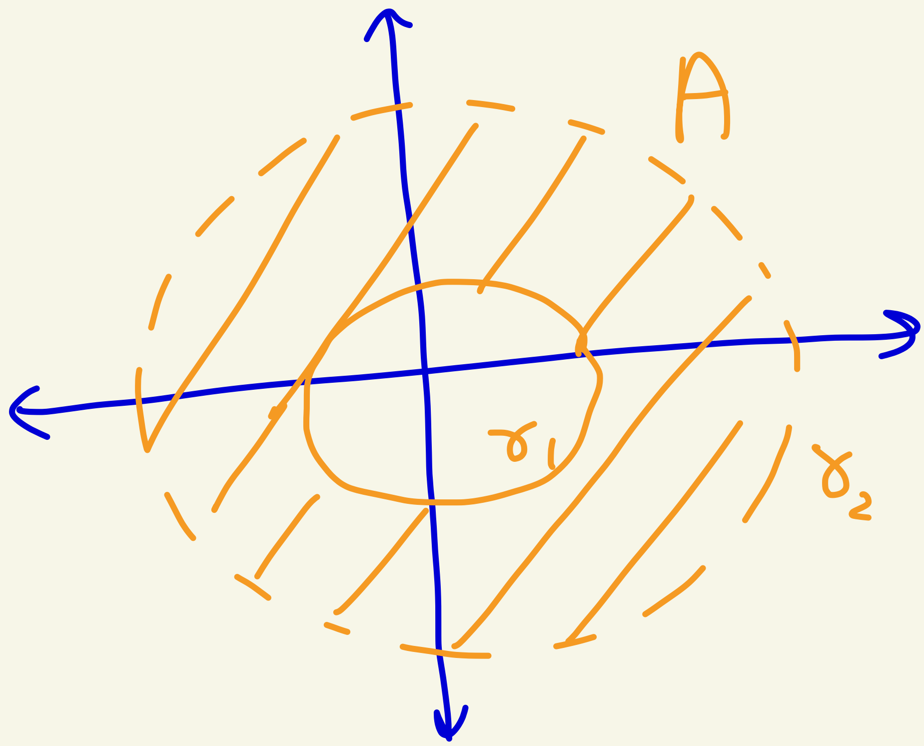
So, $|z| = 10^{1/6} > 1$.

$[10^{1/6} \simeq 1.467799\ldots]$

Thus, none of the zeros of f lie inside γ_1.

So, $p(z)$ has no zeros inside γ_1.

By part (i) and (ii), all of the zeros of $p(z) = z^6 - 5z^2 + 10$ lie in $A = \{z \mid 1 \le |z| < 2\}$

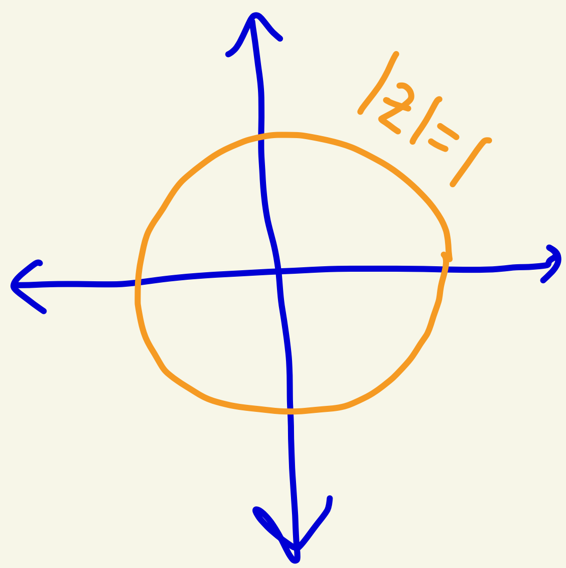


② We are interested in the zeros of the function $g(z) = e^z - cz^n$

Let $f(z) = -cz^n$ and $h(z) = e^z$.

Notice that f has a zero at $z_0 = 0$ of multiplicity n. And f has no other zeros in $\mathbb{C}$.

Let γ_1 be the curve $|z| = 1$.



If z is on γ_1 then $|z| = 1$ and we have

$$|f(z)| = |-cz^n| = |c||z|^n = c \cdot 1^n = c > e$$

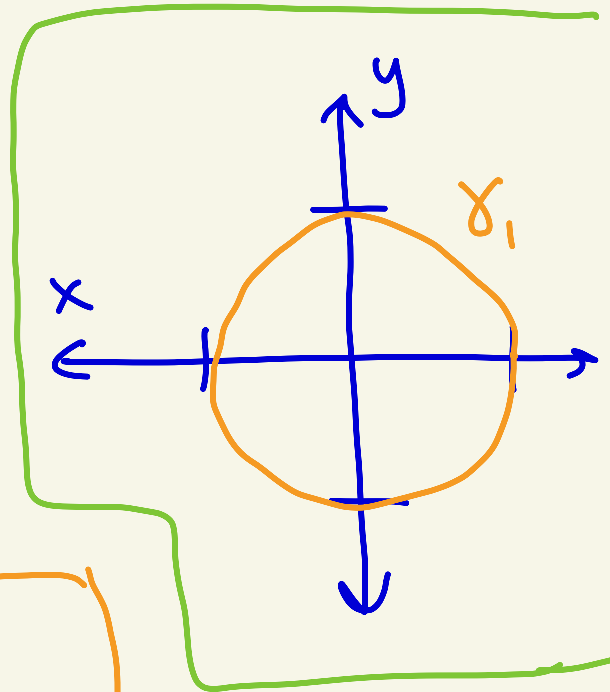
$c \in \mathbb{R}$
$c > e$

and $(z = x + iy)$

$$h(z) = |e^z| = |e^x e^{iy}|$$

$$= |e^x|\underbrace{|e^{iy}|}_{1} = |e^x|$$

$$= e^x \leq e$$



$e^x > 0$

$|z| = 1$
so $-1 \leq x \leq 1$

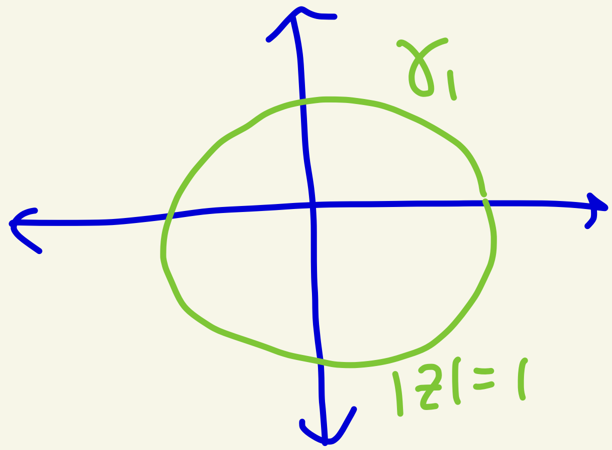
Thus, if z is on γ_1, then

$$|h(z)| \leq e < |f(z)|.$$

So, by Rouché's theorem, $f(z) = -cz^n$
and $g(z) = h(z) + f(z) = e^z - cz^n$
both have n zeros (counting multiplicity)
in $|z| < 1$ [ie inside of γ_1]

③ We want to show that the function $p(z) = g(z) - z$ has exactly one zero inside the unit circle.

Let γ_1 be the unit circle $|z| = 1$.



Let $f(z) = -z$.

Then, if z is on γ_1 then $|z| = 1$ and
$$|g(z)| < 1 = |z| = |-z| = |f(z)|$$

So, by Rouché's theorem, both $f(z) = -z$ and $p(z) = g(z) + f(z) = g(z) - z$ have the same number of zeroes (counting multiplicity) inside of γ_1.

Since $f(z) = -z$ has 1 zero inside of γ_1, ie with $|z| < 1$, We have that $p(z) = g(z) - z$ has exactly one zero inside $|z|<1$.

Thus, $g(z) = z$ at exactly one fixed point z in $|z|<1$.