Math 5800

10/11/21

HW 4-6(b)
- removed hint on problem statement
- made two solutions for the problem

Test 1

Monday 10/18

Study - notes & homework

Hw 3 -

- Does a set have measure zero or not?
- $f=g$ almost everywhere
- proofs involving measure zero
- proofs involving $f=g$ almost everywhere

Hw 4 -

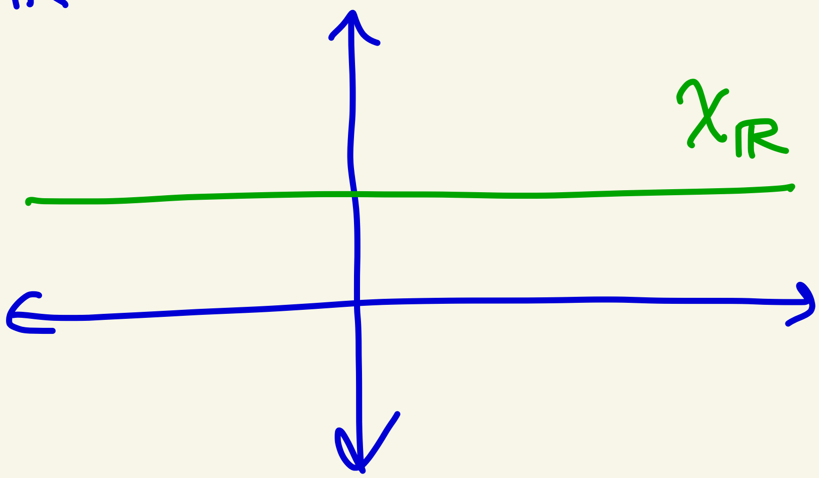
- Write step function in a rep. with only disjoint intervals
- $\int f$ for step functions f
- proofs involving characteristic functions χ_S
- proofs with step functions

Ended Topic 4 on 9/27/21

Questions:

Not every χ_S is a step function.

$\chi_{\mathbb{R}}$ is not a step function.



Step function:

$$f = c_1 \chi_{I_1} + \dots + c_n \chi_{I_n}$$

I_j are bounded intervals

can assume $c_j \neq 0$

Our goal now is to show that the def of $\int f$ when $f \in L^0$ is well-defined.

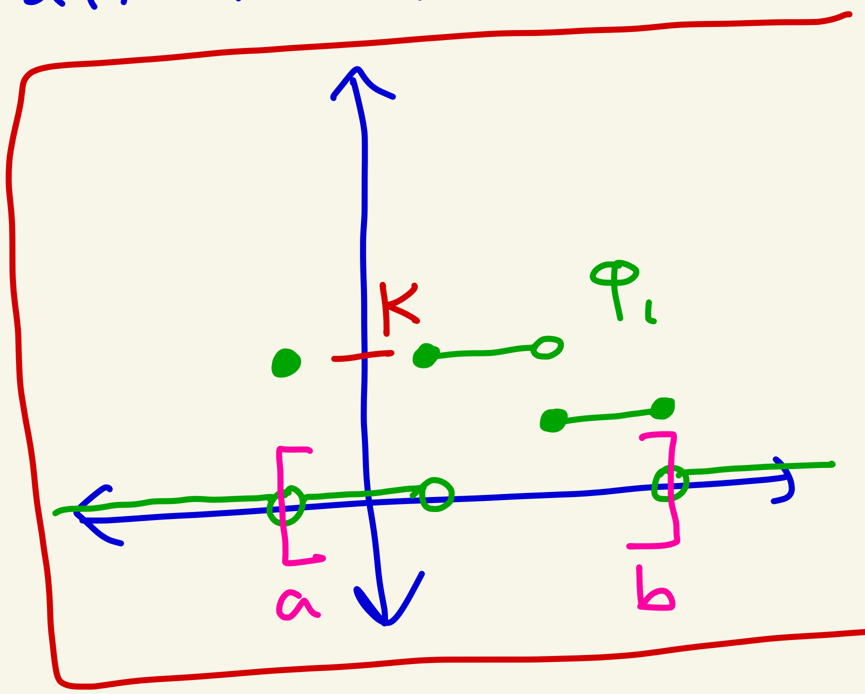
<u>Lemma:</u> Let $(\varphi_n)_{n=1}^{\infty}$ be a non-increasing sequence $\left[\varphi_{n+1}(x) \leq \varphi_n(x) \text{ for all } n \geq 1 \text{ and } x \in \mathbb{R}\right]$ of non-negative $\left[\varphi_n(x) \geq 0 \ \forall x \in \mathbb{R},\ n \geq 1\right]$ step functions such that $\lim_{n\to\infty} \varphi_n(x) = 0$ for almost all x.

Then, $\lim_{n\to\infty} \int \varphi_n = 0$.

proof:

Let $\varepsilon > 0$.

Since φ_1 is a step function, there exists an interval $[a,b]$ where $\varphi_1(x) = 0$ for all $x \in \mathbb{R} - [a,b]$



Also, since φ_1 is a non-negative step function there exists $K > 0$ where

$$0 \leq \varphi_1(x) \leq K$$

for all $x \in [a,b]$.

Since $(\varphi_n)_{n=1}^{\infty}$ is a non-increasing, non-negative sequence of step functions we know $0 \leq \varphi_n(x) \leq \varphi_1(x)$ for all $x \in \mathbb{R}$.

So, for all $n \geq 1$ we have

$\varphi_n(x) = 0$ for all $x \in \mathbb{R} - [a,b]$

and

$0 \leq \varphi_n(x) \leq K$ for all $x \in [a,b]$.

Each φ_n has a finite number of discontinuities.

Thus,

$$A = \left\{ x \;\middle|\; \begin{array}{l} \text{there exists } n \geq 1 \text{ where} \\ \varphi_n \text{ is discontinuous at } x \end{array} \right\}$$

So, A contains all the points where the φ_n's are discontinuous.

A is countable since its the countable union of finite sets.

Thus, A has measure zero.

Let

$$B = \{ x \mid \lim_{n \to \infty} \varphi_n(x) \neq 0 \}.$$

By assumption, B has measure zero.

Let $C = A \cup B$.

Then, C has measure zero.

Thus, there exists a sequence

$$I_1, I_2, I_3, \ldots$$

of bounded open intervals where

$$C \subseteq \bigcup_{j=1}^{\infty} I_j$$

and $\displaystyle\sum_{j=1}^{\infty} \ell(I_j) \leq \varepsilon$

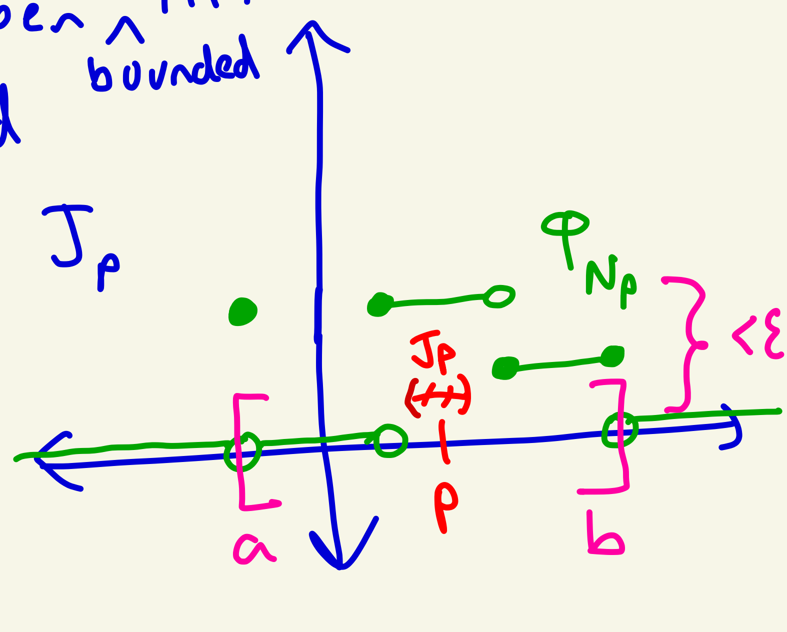
Consider any point $p \in [a,b]$ where $p \notin C$.

Then, $\lim_{n\to\infty} \varphi_n(p) = 0$

Thus, there exists an integer N_p, depending on p, where $|\varphi_{N_p}(p) - 0| < \varepsilon$.

Thus, $0 \le \varphi_{N_p}(p) = |\varphi_{N_p}(p)| < \varepsilon$.

Since $p \notin C$, it is not a point of discontinuity of φ_{N_p}, thus there must exist an open bounded interval J_p where $p \in J_p$ and φ_{N_p} is constant on J_p



Thus, $0 \le \varphi_{N_p}(x) < \varepsilon$

for all $x \in J_p$.

Also, since the sequence is non-increasing, if $n \geq N_p$

then

$$\boxed{0 \le \varphi_n(x) \le \varphi_{N_p}(x) < \varepsilon}$$

for all $x \in J_p$

The open intervals I_n with $n \geq 1$ and J_p with $p \notin C$, $p \in [a,b]$ form a open cover for $[a,b]$.

By the Heine-Borel theorem [Math 4650] there is a finite subcover

$$I_{n_1}, I_{n_2}, \ldots, I_{n_r}, J_{p_1}, J_{p_2}, \ldots, J_{p_s}$$

that cover $[a,b]$.

There may not be any J_p's in the above, ie $p_s = 0$. If that happens just add some in so we get $p_s \geq 1$.

Define $M = \max\{N_{p_1}, N_{p_2}, \ldots, N_{p_s}\}$

Then, $0 \leq \varphi_n(x) < \varepsilon$ for all $x \in \bigcup_{j=1}^{s} J_{p_j}$ when $n \geq M$.

Let

$$S = \bigcup_{i=1}^{r} \left(I_{n_i} \cap [a,b] \right)$$

and

$$T = \bigcup_{j=1}^{s} \left(J_{p_j} \cap [a,b] \right)$$

each of these is a bounded interval but might not be open

Since each S and T are the union of a finite # of bounded intervals, by HW 4 problem 7(b), we may express S and T as the union of a finite number of disjoint bounded intervals,

$$S = \bigcup_{i=1}^{a} S_i \quad \text{and} \quad T = \bigcup_{j=1}^{b} T_j$$

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Since $S \subseteq \bigcup_{i=1}^{r} I_{n_i}$

by HW 4, problem 7(e), we have

that $\sum_{i=1}^{a} \ell(S_i) \leq \sum_{i=1}^{r} \ell(I_{n_i})$

$$\leq \sum_{n=1}^{\infty} \ell(I_n) \leq \varepsilon.$$

Since $T \subseteq [a,b]$, by HW 7

problem 12, we know

$$\sum_{j=1}^{b} \ell(T_j) \leq b-a$$

By all the above, if $x \in [a,b]$, then

$$\varphi_n(x) \leq K \cdot \chi_S(x) + \varepsilon \cdot \chi_T(x)$$

for all $n \geq M$.

if $x \in S$, then $S \subseteq [a,b]$,
so $\varphi_n(x) \leq K$
if $x \in T$, then $T \subseteq \bigcup J_{e_j}$
so $\varphi_n(x) < \varepsilon$

Note, $S, T \in \mathcal{R}$ from HW, so χ_S and χ_T are step functions.

Thus, integrating the previous formula we get if $n \geq M$ then

$$0 \leq \int \varphi_n \leq \int K \cdot \chi_S + \varepsilon \cdot \chi_T$$

$$= K \cdot \sum_{i=1}^{a} \ell(S_i) + \varepsilon \cdot \sum_{j=1}^{b} \ell(T_j)$$

$$\leq K \cdot \varepsilon + \varepsilon \cdot (b-a)$$

$$= \varepsilon \left[K + (b-a) \right]$$

Thus, given $\varepsilon' > 0$ we can set $\varepsilon = \frac{1}{K+(b-a)} \cdot \frac{\varepsilon'}{2}$ and there will exist an $M > 0$ where if $n \geq M$, then

$$\left| \int \varphi_n - 0 \right| = \int \varphi_n \leq \varepsilon \left[K + (b-a) \right] = \frac{\varepsilon'}{2} < \varepsilon'$$

Thus $\lim_{n \to \infty} \int \varphi_n = 0$. $\blacksquare$