MATH 5800

11/1/21

Grading schemes

syllabus

test 1 - $1/3$
test 2 - $1/3$
final - $1/3$

drop one

$\max\{\text{test } 1, \text{test } 2\}$ - $1/2$
final - $1/2$

I will pick the better of "syllabus" or "drop one" method for each student.

Test 2

Monday Nov 15

Covers HW 6 and HW 7

Topic 8 – More on integrable functions

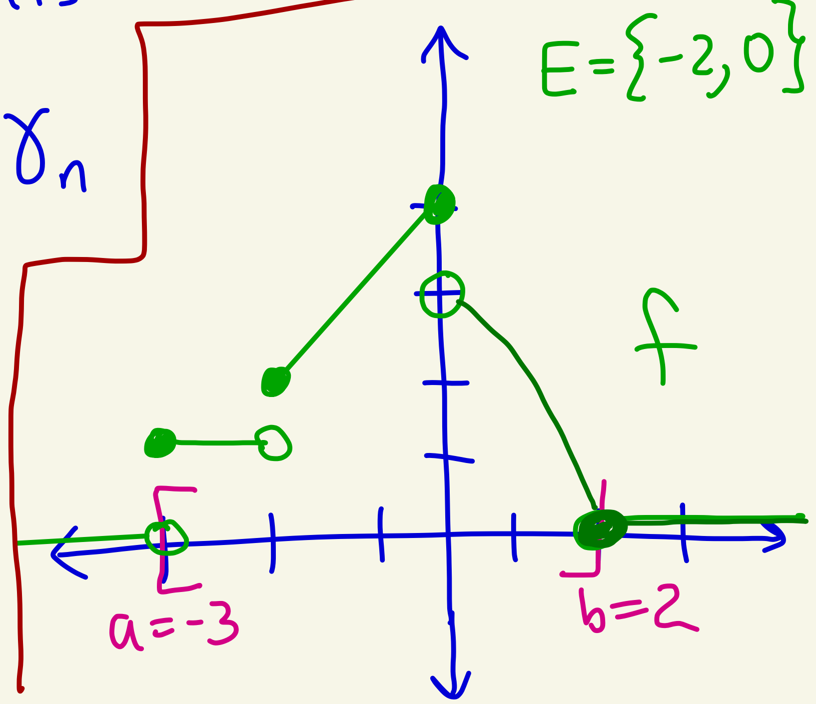
Theorem: Let $f:\mathbb{R}\to\mathbb{R}$ be bounded on $[a,b]$ and vanish outside of $[a,b]$ [That is, $f(x)=0$ for all $x\notin[a,b]$].

Let
$$E=\{x\in(a,b)\mid f \text{ is discontinuous at } x\}$$

If E has measure zero, then f is integrable [indeed, we will show $f\in L^0$]

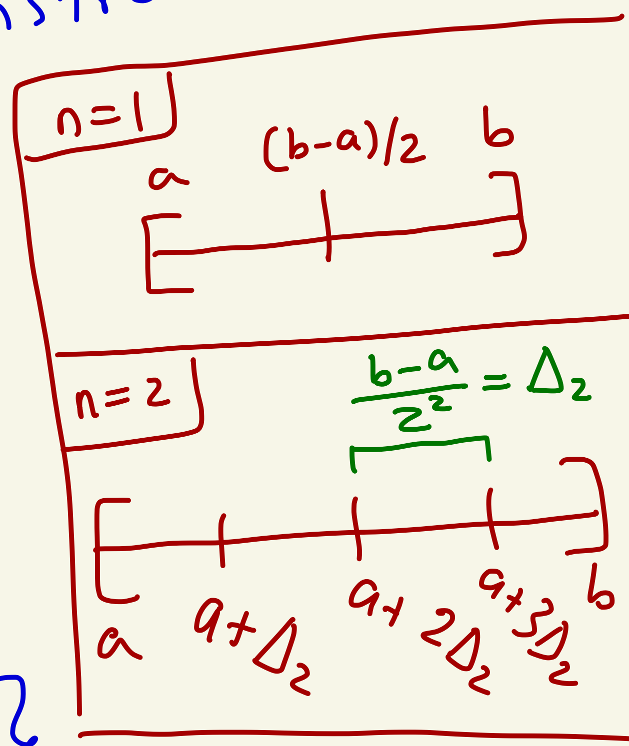
Furthermore, if this is the case then
$$\int f=\lim_{n\to\infty}\int \gamma_n$$

where γ_n is the standard construction on $[a,b]$.



proof: Suppose E has measure zero.

Let $E' \subseteq [a,b]$ consist of all the endpoints of all the $I_{n,k}$ for every subdivision of $[a,b]$ into 2^n subintervals via the standard construction.



So,

$$E' = \left\{ a, b, \frac{b-a}{2}, a + \frac{b-a}{2^2}, a + 2\cdot\frac{b-a}{2^2}, a + 3\cdot\frac{b-a}{2^2}, a + \frac{b-a}{2^3}, a + 2\cdot\frac{b-a}{2^3}, \dots \right\}$$

Then, E' is countable and so E' has measure zero.

Let $F = E \cup E'$.

Then F has measure zero.

Let $(\gamma_n)_{n=1}^{\infty}$ be the standard construction for f on $[a,b]$.

$$\boxed{\underline{\text{Claim 1:}}\ \left(\int \gamma_n\right)_{n=1}^{\infty} \text{ is bounded}}$$

<u>proof of claim 1:</u>

Since f is bounded on $[a,b]$ there exists $K > 0$ where

$$\underbrace{|f(x)| \leq K}_{-K \leq f(x) \leq K} \text{ for all } x \in [a,b].$$

Let $m = 2^n$. Then,

$$\gamma_n = c_{n,1}\chi_{I_{n,1}} + c_{n,2}\chi_{I_{n,2}} + \dots + c_{n,m}\chi_{I_{n,m}}$$

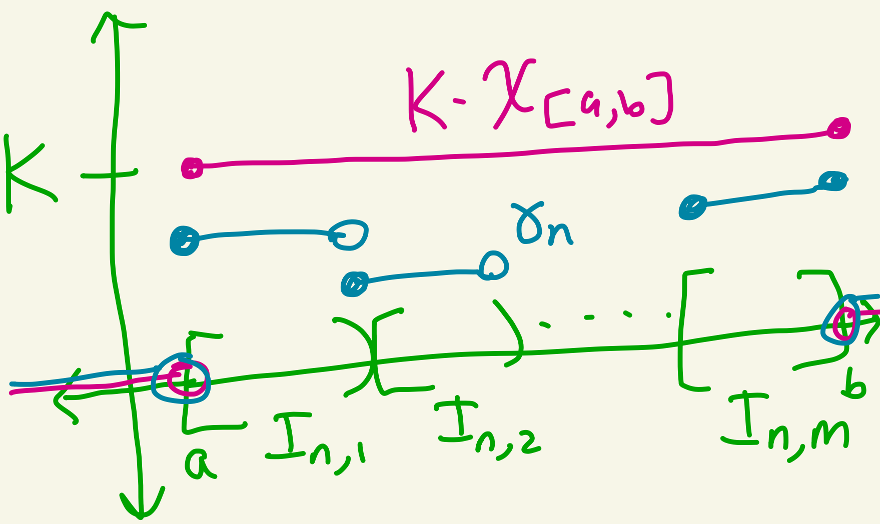
where $c_{n,k} = \inf\{f(t) \mid t \in I_{n,k}\} \leq K$

Thus,

$$\gamma_n \leq K \cdot \chi_{I_{n,1}} + K \cdot \chi_{I_{n,2}} + \dots + K \cdot \chi_{I_{n,m}}$$

$$= K \cdot \chi_{[a,b]}$$

because $[a,b]$ is the disjoint union of $I_{n,1}, I_{n,2}, \dots, I_{n,m}$.



Thus,

$$\int \gamma_n \leq \int K \cdot \chi_{[a,b]}$$

$$= K \cdot (b-a).$$

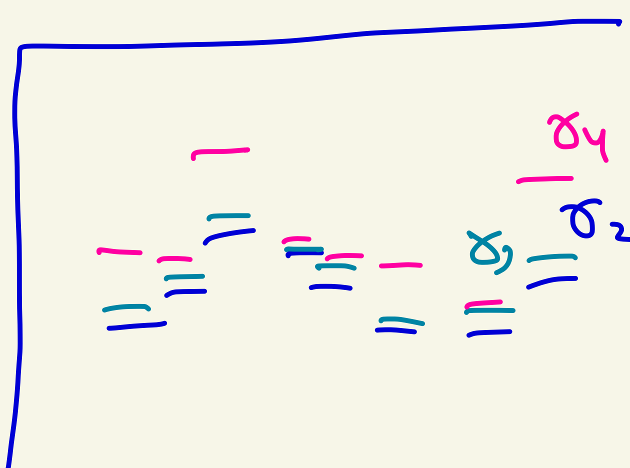
$\boxed{\text{Claim I}}$

Since $(\gamma_n)_{n=1}^{\infty}$ is a non-decreasing sequence of step functions with $\left(\int \gamma_n\right)_{n=1}^{\infty}$ bounded we know that $(\gamma_n(x))_{n=1}^{\infty}$ converges for almost all x.

We want to show that $\gamma_n \to f$ for almost all x.

Since $\gamma_n(x) = 0 = f(x)$ for all $x \notin [a,b]$.

So, $\gamma_n(x) \to f(x)$ for all $x \notin [a,b]$.



Let $p \in [a,b] - F$.

$\boxed{\text{Claim 2: } \gamma_n(p) \to f(p) \text{ as } n \to \infty}$

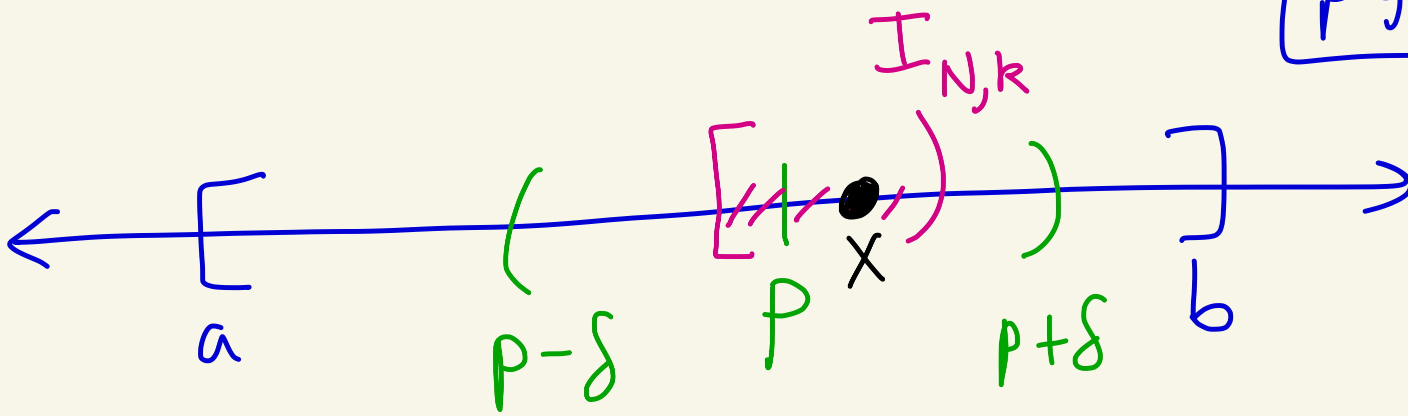
<u>proof of claim 2:</u>

Let $\varepsilon > 0$

Since $p \notin E$ we know that f is continuous at p.

Thus, there exists $\delta > 0$ where if $|x-p| < \delta$, then $|f(x) - f(p)| < \frac{\varepsilon}{2}$

<u>Note:</u> Since $p \notin F$ we know $p \neq a$ and $p \neq b$. Thus we may pick δ small enough so that $(p-\delta, p+\delta) \subseteq (a,b)$ which will ensure that if $|x-p| < \delta$ then $x \in (a,b)$.



Choose $N > 0$ where $\frac{b-a}{2^N} < \delta$.

Let $I_{N,k}$ be the sub-interval that p is in on the N-th subdivision of $[a,b]$ in the standard construction.

Since $\ell(I_{N,k}) = \frac{b-a}{2^N} < \delta$, we have that $|x-p| < \delta$ for all $x \in I_{N,k}$.

So, $|f(x) - f(p)| < \frac{\varepsilon}{2}$
for all $x \in I_{N,k}$.

Thus, $f(p) - \frac{\varepsilon}{2} < f(x) < f(p) + \frac{\varepsilon}{2}$
for all $x \in I_{N,k}$

$\boxed{z > 0}$

$|x-y| < z$
$y - z < x < y + z$

Recall that
$\gamma_N(p) = \inf\{f(t) \mid t \in I_{N,k}\}$

Since $f(p) - \frac{\varepsilon}{2}$ is a lower bound
on $\{f(t) \mid t \in I_{N,k}\}$ we know
that $f(p) - \frac{\varepsilon}{2} \leq \gamma_N(p)$.

Also, $\gamma_N(p) \leq f(p)$.

$\gamma_n(x) \leq f(x)$
always

Thus,
$$f(p) - \varepsilon < f(p) - \frac{\varepsilon}{2} \leq \gamma_N(p) \leq f(p) < f(p) + \varepsilon$$

Thus,

$$f(p)-\varepsilon < \gamma_N(p) < f(p)+\varepsilon.$$

Since $(\gamma_n)_{n=1}^{\infty}$ is non-decreasing we know that if $n \geq N$ then $\gamma_N(p) \leq \gamma_n(p)$.

Also, $\gamma_n(p) \leq f(p)$ for all n.

So, if $n \geq N$, then

$$f(p)-\varepsilon < \gamma_N(p) \leq \gamma_n(p) \leq f(p) < f(p)+\varepsilon$$

So if $n \geq N$, then

$$f(p)-\varepsilon < \gamma_n(p) < f(p)+\varepsilon$$

Thus, if $n \geq N$, then

$$|\gamma_n(p)-f(p)| < \varepsilon$$

Claim 2

Summarizing, $(\gamma_n)_{n=1}^{\infty}$ is a non-decreasing sequence of step functions and $\left(\int \gamma_n\right)_{n=1}^{\infty}$ is a bounded sequence.

And $\gamma_n(x) \to f(x)$ for all $x \notin F$.

Since F has measure zero, $\gamma_n \to f$ almost everywhere in $\mathbb{R}$.

Thus, $f \in L^0$ and $\int f = \lim_{n\to\infty} \int \gamma_n$

So, $f \in L^1$ and is integrable.

$\blacksquare$