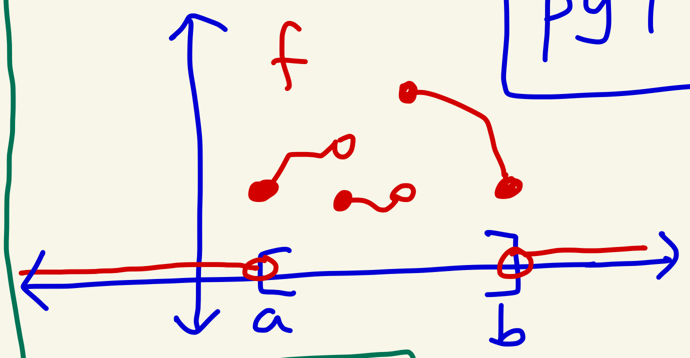
Math 5800

11/3/21

Topic 8 continued...



Recap from last time

Theorem: Let $f : \mathbb{R} \to \mathbb{R}$ be bounded on $[a,b]$ and vanishes outside of $[a,b]$. Let $E = \{ x \in (a,b) \mid f \text{ is discontinuous at } x \}$

If E has measure zero, then $f \in L^0$ and so $f \in L^1$.

And
$$\int f = \lim_{n \to \infty} \int \gamma_n$$

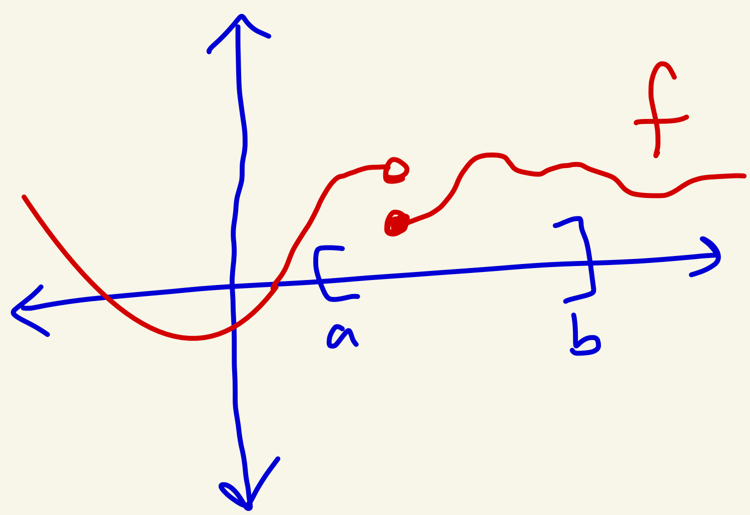
where γ_n is the standard construction for f on $[a,b]$.

Corollary: If $f: \mathbb{R} \to \mathbb{R}$ and f is bounded on $[a,b]$ and

$$E = \{x \in (a,b) \mid f \text{ is discontinuous at } x\}$$

has measure zero, then

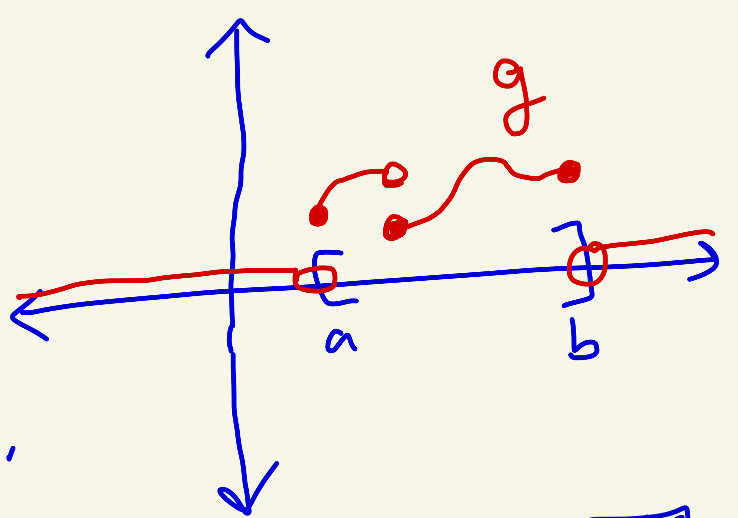
$$f \in L^1([a,b]).$$



proof:

Let

$$g = f \cdot \chi_{[a,b]}$$



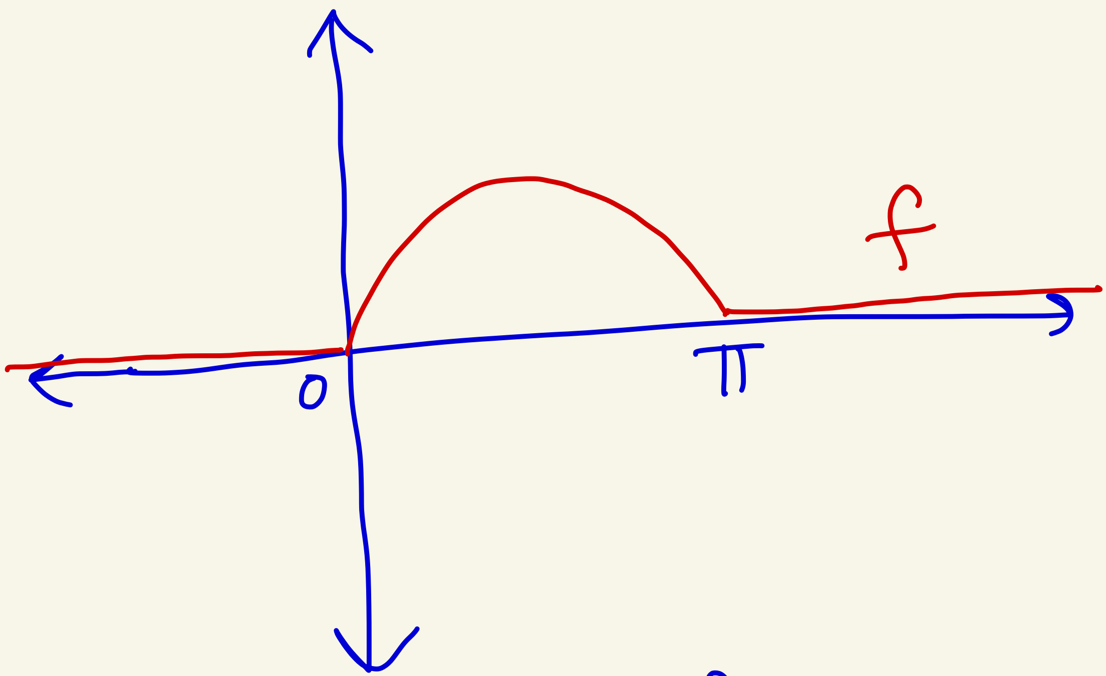
By the previous theorem $g \in L^1$.

Thus, $f \in L^1([a,b])$.

$\blacksquare$

<u>Ex:</u> Let

$$f(x) = \chi_{[0,\pi]}(x) \cdot \sin(x) = \begin{cases} \sin(x) & \text{if } 0 \le x \le \pi \\ 0 & \text{otherwise} \end{cases}$$



$f \in L^0$ and $\int f = \lim_{n\to\infty} \int \gamma_n$

where γ_n is the standard construction on $[0,\pi]$.

Theorem: Let $f: \mathbb{R} \to \mathbb{R}$ be bounded on $[a,b]$.
Then f is $\underbrace{\text{Riemann integrable}}_{\text{(Math 4660)}}$ on $[a,b]$ if and only if

$$E = \{ x \in (a,b) \mid f \text{ is discontinuous at } x \}$$

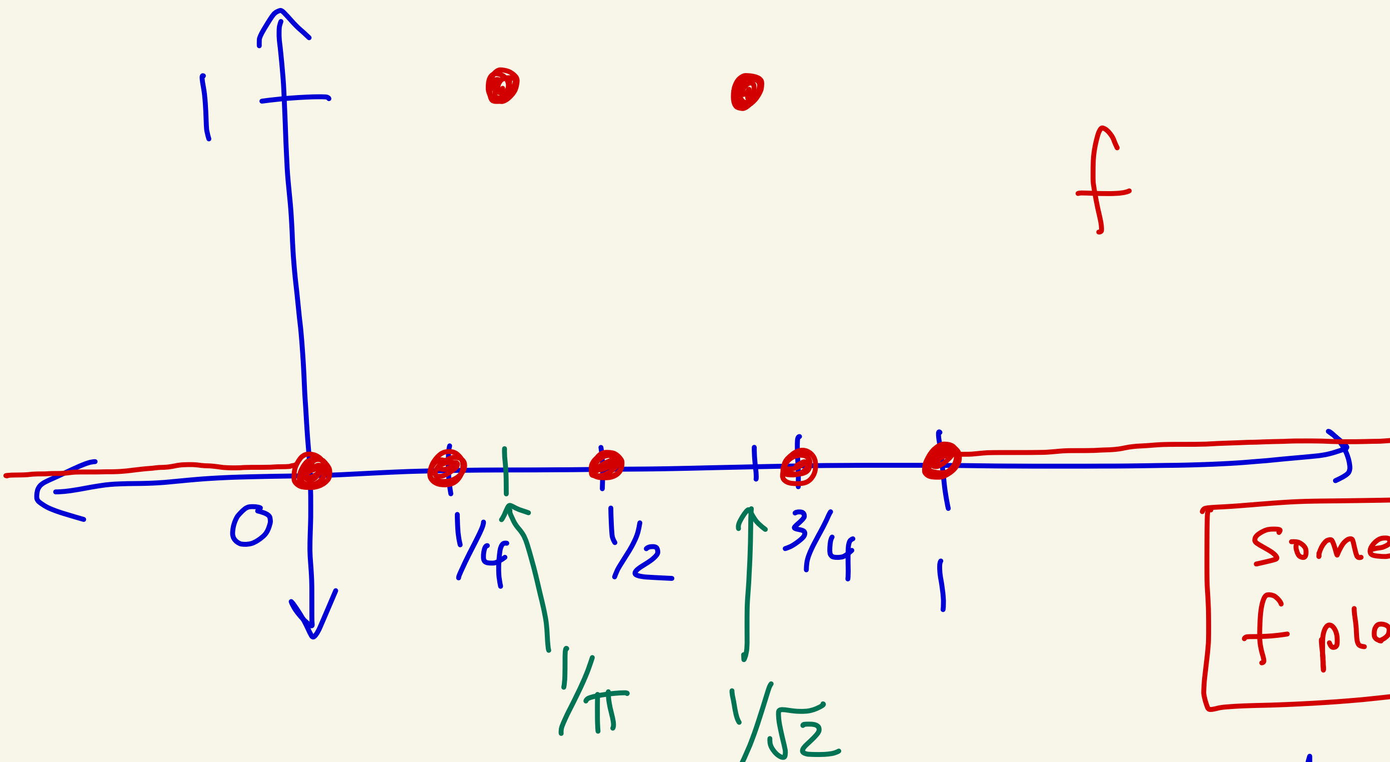
has measure zero.

Furthermore, if f is Riemann integrable on $[a,b]$ then f is Lebesgue integrable on $[a,b]$ (ie $f \in L^1([a,b])$) and

$$\underbrace{\int_{[a,b]} f}_{\text{Lebesgue integral on } [a,b]} = \underbrace{\int_a^b f(x)\,dx}_{\text{Riemann integral on } [a,b]}$$

Ex: Let

$$f(x) = \begin{cases} 0 & \text{if } x \notin [0,1] \\ 1 & \text{if } x \text{ is irrational and } x \in [0,1] \\ 0 & \text{if } x \text{ is rational and } x \in [0,1] \end{cases}$$



Some of f plotted

$E = \{ x \in (0,1) \mid f \text{ is discontinuous at } x \}$

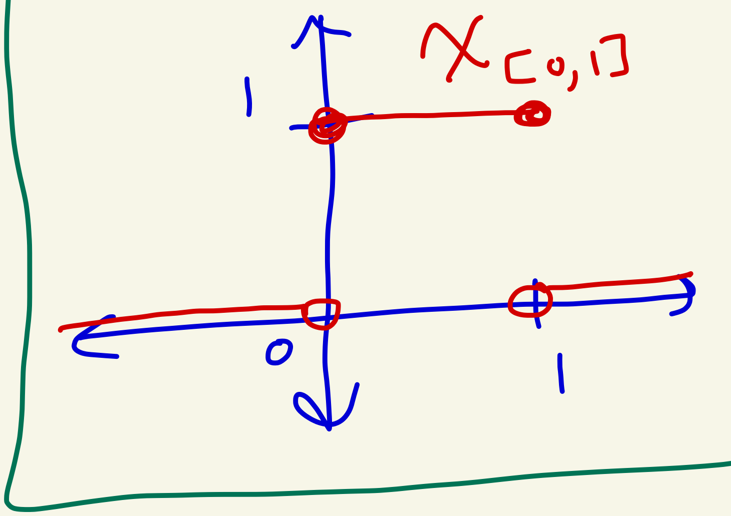
$= (0,1)$ which doesn't have measure zero

So, f is not Riemann integrable on $[0,1]$.

Note that

$$\chi_{[0,1]}(x) = f(x)$$

everywhere except at $\mathbb{Q} \cap [0,1]$.



And $\mathbb{Q} \cap [0,1]$ has measure zero, because $\mathbb{Q} \cap [0,1] \subseteq \mathbb{Q}$ and $\mathbb{Q}$ has measure zero.

So, $\chi_{[0,1]} = f$ almost everywhere.

Consider the constant sequence $\varphi_n = \chi_{[0,1]}$ for all $n \geq 1$, ie

$$\chi_{[0,1]}, \chi_{[0,1]}, \chi_{[0,1]}, \ldots$$

$$\varphi_1, \quad \varphi_2, \quad \varphi_3, \ldots$$

Then, $\lim_{n\to\infty} \varphi_n(x) = \chi_{[0,1]}(x) = f(x)$

for all $x \notin \mathbb{Q} \cap [0,1]$.

So, $\varphi_n \longrightarrow f$ almost everywhere

and $(\varphi_n)_{n=1}^{\infty}$ is a non-decreasing sequence and $\int \varphi_n = \int \chi_{[0,1]} = 1$

and so $(\int \varphi_n)_{n=1}^{\infty}$ is a bounded sequence.

Thus, $f \in L^0$.

So, $f \in L^1$ and is Lebesgue integrable and

$$\int f = \lim_{n\to\infty} \int \varphi_n = 1.$$

So, f is not Riemann integrable on $[0,1]$ but it is Lebesgue integrable on $[0,1]$. $\blacksquare$