Math 5800

12/8/21

# Final exam

- Weds Dec 15
- opens at 5am on Weds 12/15 and closes at 12pm noon on Thursday 12/16.
- You will get a 3 hr window to take the exam

- covers:
  - Test 1 material
  - Test 2 material
  - HW 8
  - HW 9

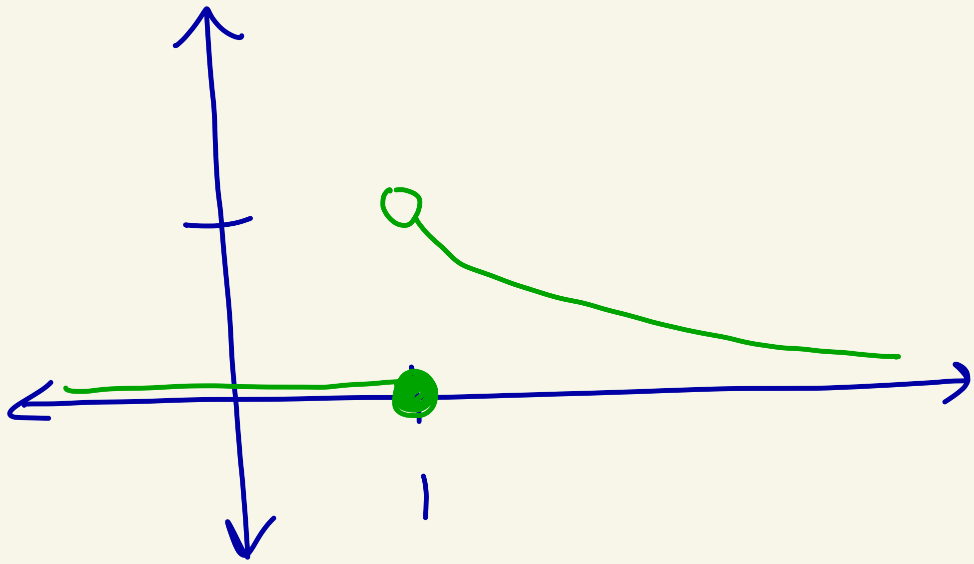
- I emailed out a more thourough study guide

# HW 8

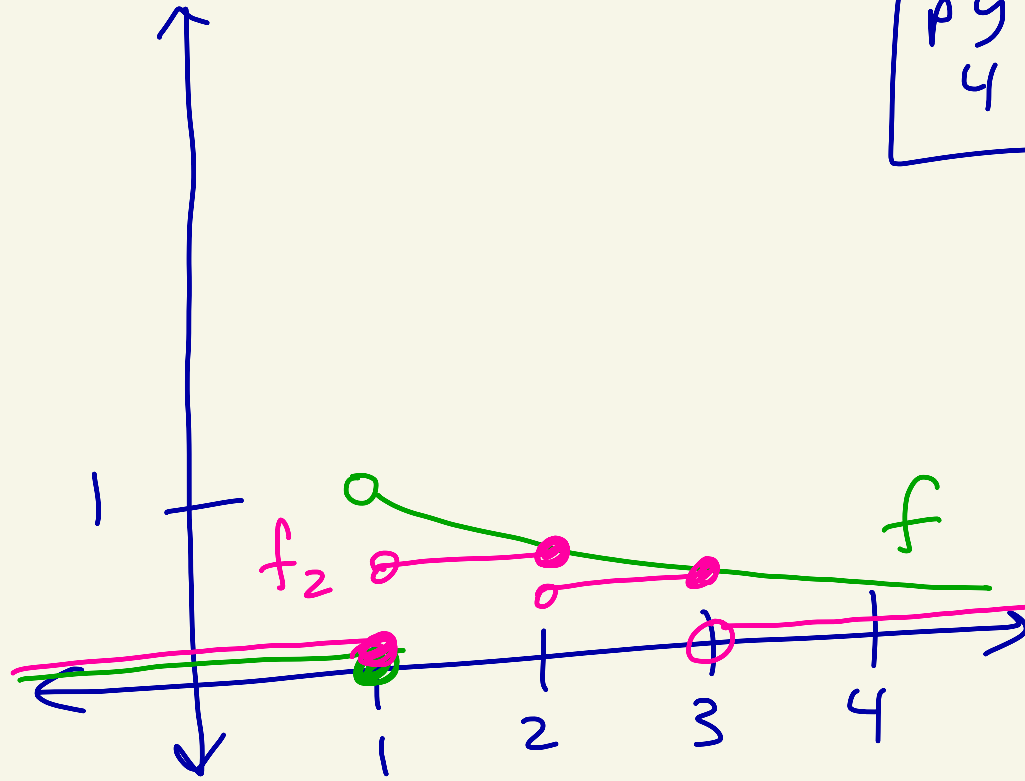
① $f: \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} 0 & \text{if } x \leq 1 \\ 1/x & \text{if } x > 1 \end{cases}$$

Show $f \notin L^1$




Method: Find a sequence $\{f_n\}_{n=1}^{\infty}$ of $L^1$ functions with $f_n \leq f$ and $\int f_n \to \infty$.




Define

$$f_n = \frac{1}{2}\cdot \chi_{(1,2]} + \frac{1}{3}\cdot \chi_{(2,3]} + \dots + \frac{1}{n}\cdot \chi_{(n-1,n]}$$

Then, $f_n(x) \le f(x)$ for all $x \in \mathbb{R}$ and $n \geq 1$.

Note $f_n$ is a step function for each $n \geq 1$, so $f_n \in L^1$ for $n \geq 1$.

Note

$$\int f_n = \int \sum_{k=2}^{n} \frac{1}{k} \cdot \chi_{(k-1,k]}$$

$$= \sum_{k=2}^{n} \frac{1}{k} \cdot \underbrace{[k-(k-1)]}_{1} = \sum_{k=2}^{n} \frac{1}{k}$$

Suppose $f \in L^1$.

Then $\int f$ would be a finite number and since $f_n \leq f$ for all $n \geq 1$ we would have

$$\sum_{k=2}^{n} \frac{1}{k} \leq \int f.$$

$\boxed{\sum_{k=1}^{\infty} \frac{1}{k} \text{ diverges} \quad \text{harmonic series}}$

But $\lim_{n\to\infty} \sum_{k=2}^{n} \frac{1}{k} = \infty$

But then $\int f$ would be infinite.

Contradiction. So, $f \notin L^1$. $\blacksquare$

PS 6

Don't worry about 5(a)
Just assume result is true.
Can look at Weir doc
I sent to see how to
adjust the proof.

8



(a) Let $f$ and $h$ be measurable functions.
Show $f+h$ is a measurable function.

<u>proof:</u>

Let $f$ and $h$ be measurable functions.
Since $f$ is measurable there exists a sequence of $L^1$ functions $(f_n)_{n=1}^{\infty}$ where $f_n \to f$ on an almost everywhere set $F \subseteq \mathbb{R}$.
Since $g$ is measurable there exists a sequence of $L^1$ functions $(g_n)_{n=1}^{\infty}$ where $g_n \to g$ on almost everywhere set $G \subseteq \mathbb{R}$.

Note $F \cap G$ is an almost everywhere set.

Thus, if $x \in F \cap G$ then

$$\lim_{n \to \infty} \left(f_n(x) + g_n(x)\right) = f(x) + g(x)$$

$f_n(x) \to f(x)$ since $x \in F$
$g_n(x) \to g(x)$ since $x \in G$

So, $f_n + g_n \to f + g$ almost everywhere.

Since $f_n, g_n \in L^1$ for all $n \geq 1$, we know $f_n + g_n \in L^1$ for all $n \geq 1$.

Thus, $(f_n + g_n)_{n=1}^{\infty}$ is a sequence of $L^1$ functions converging almost everywhere to $f+g$.

By Thm from class

So, $f+g$ is measurable. $\blacksquare$

8 (e)

Let $g \in L^1$,
Suppose $f$ is measurable and
$|f(x)| \leq g(x)$ for almost all $x$.

Then, $f \in L^1$.

<u>proof:</u>

Let $g \in L^1$ and $f$ is measurable
with $|f(x)| \leq g(x)$ for a.a.x.

Then, $g$ is a non-negative function.
Since $|f(x)| \leq g(x)$ for almost all $x$,
then $-g(x) \leq f(x) \leq g(x)$ for almost all $x$.

Thus, $\text{mid}\{-g(x), f(x), g(x)\} = f(x)$ for almost all $x$.

Because $f$ is measurable and $g \in L^1$ and $g \geq 0$ we know $\operatorname{mid}\{-g, f, g\} \in L^1$

Since $\operatorname{mid}\{-g, f, g\} \in L^1$ and $f = \operatorname{mid}\{-g, f, g\}$ almost everywhere, we know from class/previous HW that $f \in L^1$.

$\blacksquare$