Math 5800

8/30/21

Office hours

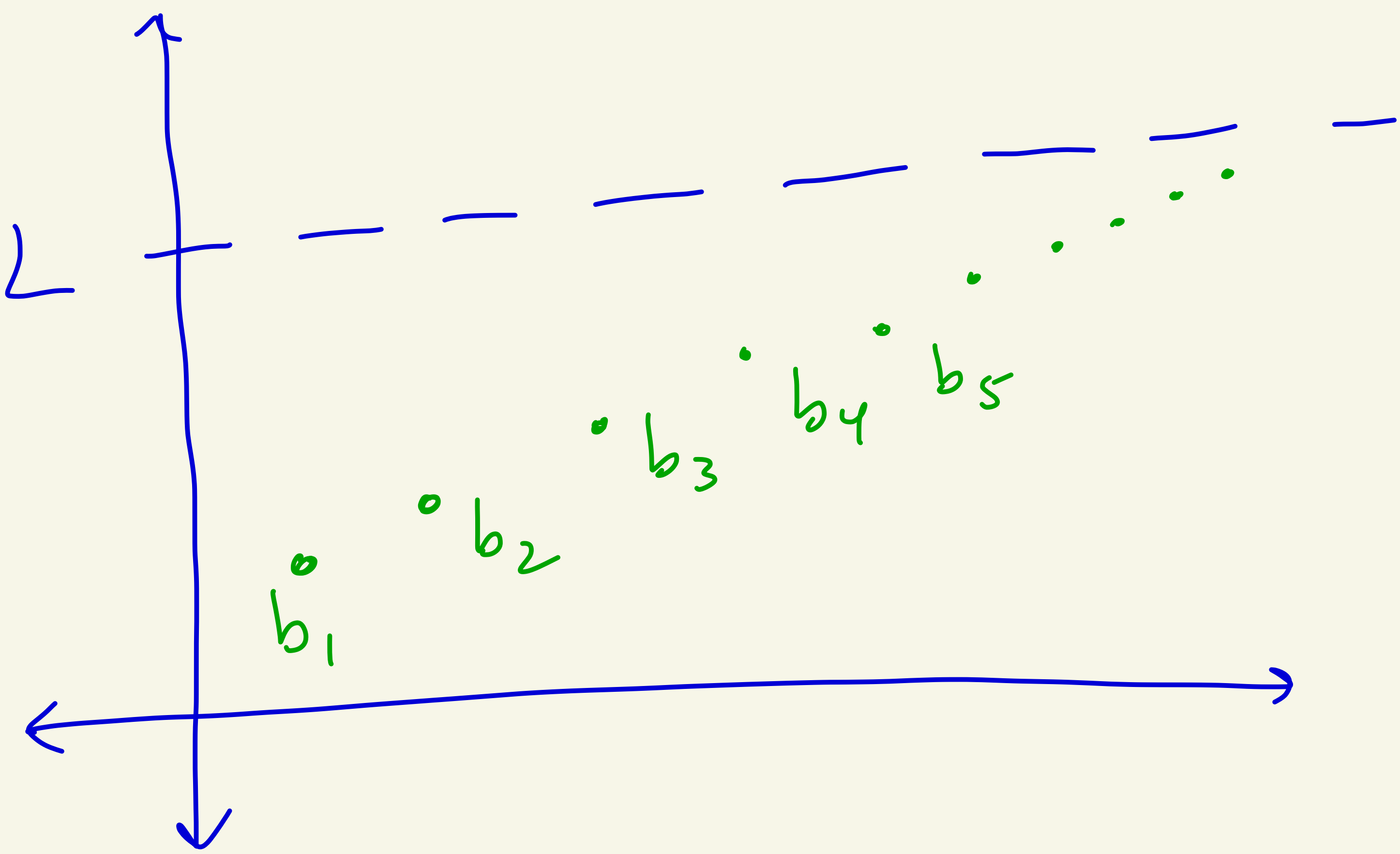
Monday 12:30 – 1:30
Tuesday 12:30 – 2:00

Zoom link is on canvas under "Office hours" page

Theorem: Let $(b_n)_{n=1}^{\infty}$ be a non-decreasing sequence of real numbers that converges to L.

Then, $b_n \leq L$ for all $n \geq 1$.

proof: HW 2 $\blacksquare$



<u>Corollary:</u> Let $\sum_{n=1}^{\infty} a_n$ be an infinite sum that converges to S and also $a_n \geq 0$ for all $n \geq 1$.

Then,

$$S_k = a_1 + a_2 + \dots + a_k \leq S.$$

<u>proof:</u>

We have $S_k = a_1 + a_2 + \dots + a_k$ are the partial sums.

Since each $a_n \geq 0$, the sequence

$$S_1, \quad S_2, \quad S_3, \dots$$

$$a_1, \quad a_1 + a_2, \quad a_1 + a_2 + a_3, \dots$$

is non-decreasing.

We are given that $\lim_{k\to\infty} s_k = S$. (pg 4

So, by the previous theorem

$$s_k \leq S \quad \text{for all } k \geq 1.$$

Topic 3 – Measure Zero

<u>Def:</u> An interval I is called a <u>bounded interval</u> if I is of the form (a,b), $(a,b]$, $[a,b)$, or $[a,b]$ where $a,b \in \mathbb{R}$ and $a \leq b$.

I is called an <u>unbounded interval</u> if I is of the form (a,∞), $[a,\infty)$, $(-\infty,a)$, $(-\infty,a]$, or $(-\infty,\infty)$.

If $a=b$, then $[a,b]=[a,a]=\{a\}$ and $(a,b)=(a,a)=\emptyset$ are both bounded.

Similarly,

$(a, a] = \emptyset$ is bounded

Ex:

bounded

$(1, 7]$

$[-13, 2100)$

$\emptyset$

$\{7\} = [7, 7]$

Unbounded

$\mathbb{R} = (-\infty, \infty)$

$(5, \infty)$

Def: The length of a bounded interval is defined as follows:

$$\ell((a,b)) = b-a$$
$$\ell([a,b)) = b-a$$
$$\ell((a,b]) = b-a$$
$$\ell([a,b]) = b-a$$

Ex: $\ell((5,7]) = 7-5 = 2$

$$\ell(\phi) = \ell((1,1)) = 1-1 = 0$$
$$\ell(\{7\}) = \ell([7,7]) = 7-7 = 0$$

Def: A bounded open interval is one of the form (a,b) where $a \leq b$.

Def: Let $S \subseteq \mathbb{R}$.

We say that S has <u>measure zero</u>, or that S is a <u>null set</u>, if for every $\varepsilon > 0$ there exists a sequence of bounded open intervals

$$I_1, I_2, I_3, I_4, \ldots$$

(which may be finite)

where

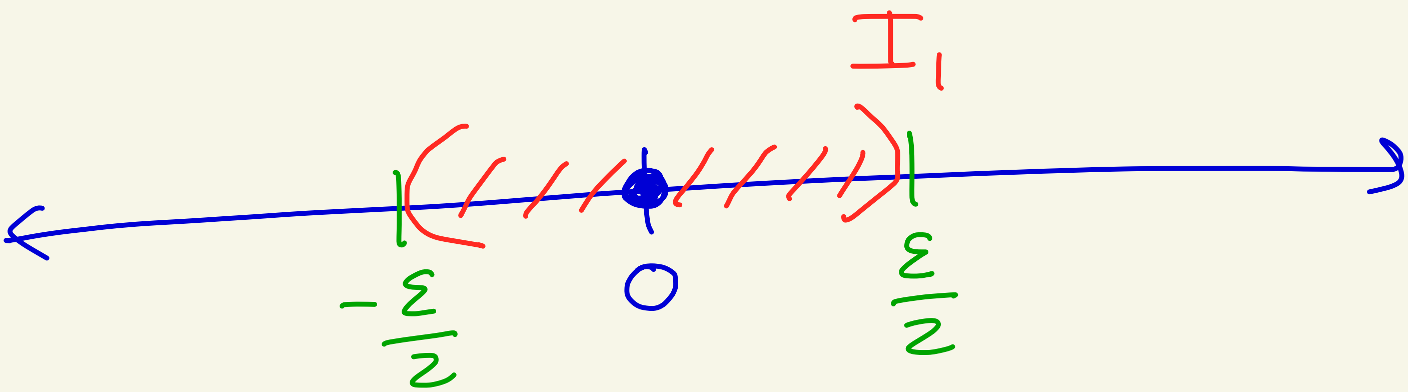
① $S \subseteq \bigcup_n I_n$

and ② $\sum_n \ell(I_n) \leq \varepsilon$.

Ex: $S = \{0\}$

Let's show that S has measure zero.

Pick some $\varepsilon > 0$.



Let $I_1 = \left(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right)$.

Then, $0 \in I_1$.

So, $S \subseteq I_1$.

So, condition ① is true.

And, $\ell(I_1) = \varepsilon \leq \varepsilon$.

So, condition ② is true.

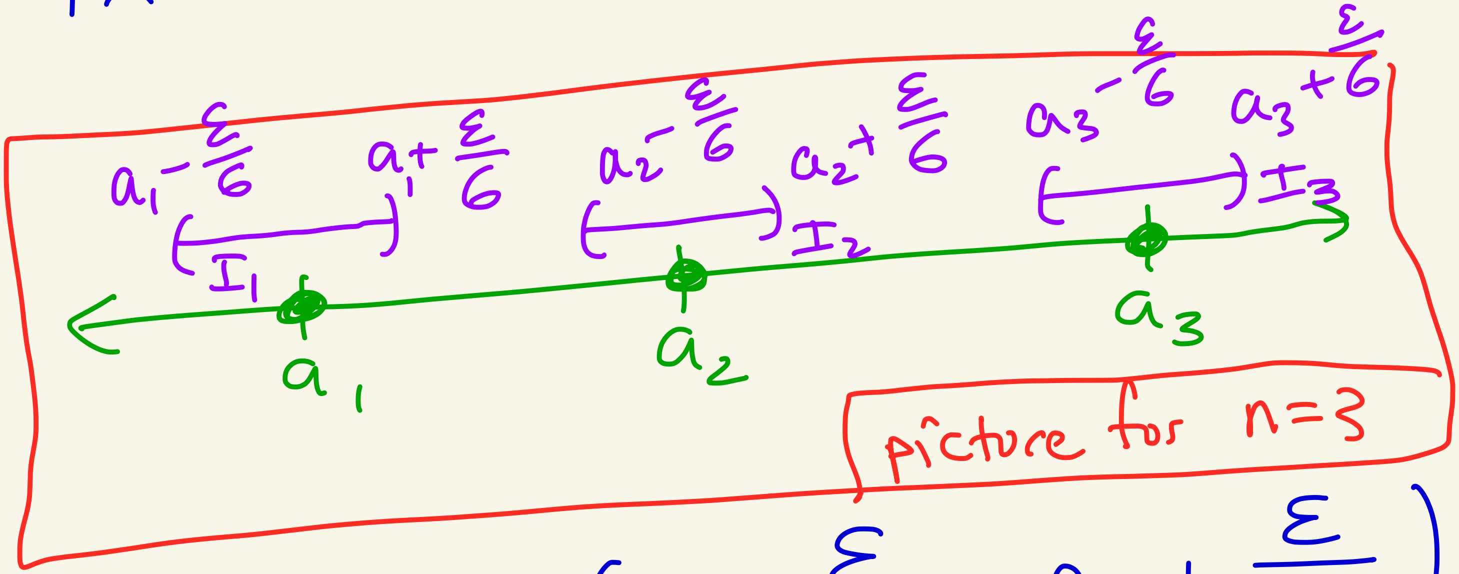
Thus, $S = \{0\}$ has measure zero.

<u>Ex:</u> (WJ book ex 1.2.1)

Let $S \subseteq \mathbb{R}$ be a finite set. Then S has measure zero.

<u>proof:</u> Let $S = \{a_1, a_2, \ldots, a_n\}$

Fix $\varepsilon > 0$.



Let $I_j = \left(a_j - \frac{\varepsilon}{2n}, a_j + \frac{\varepsilon}{2n}\right)$.

Then, $a_j \in I_j$ and

$$\ell(I_j) = \left(a_j + \frac{\varepsilon}{2n}\right) - \left(a_j - \frac{\varepsilon}{2n}\right) = \frac{\varepsilon}{n}$$

So, $S \subseteq \bigcup_{j=1}^{n} I_j$

And,

$$\sum_{j=1}^{n} \ell(I_j) = \underbrace{\frac{\varepsilon}{n} + \frac{\varepsilon}{n} + \cdots + \frac{\varepsilon}{n}}^{n \text{ of these}}$$

$$= \varepsilon \le \varepsilon.$$

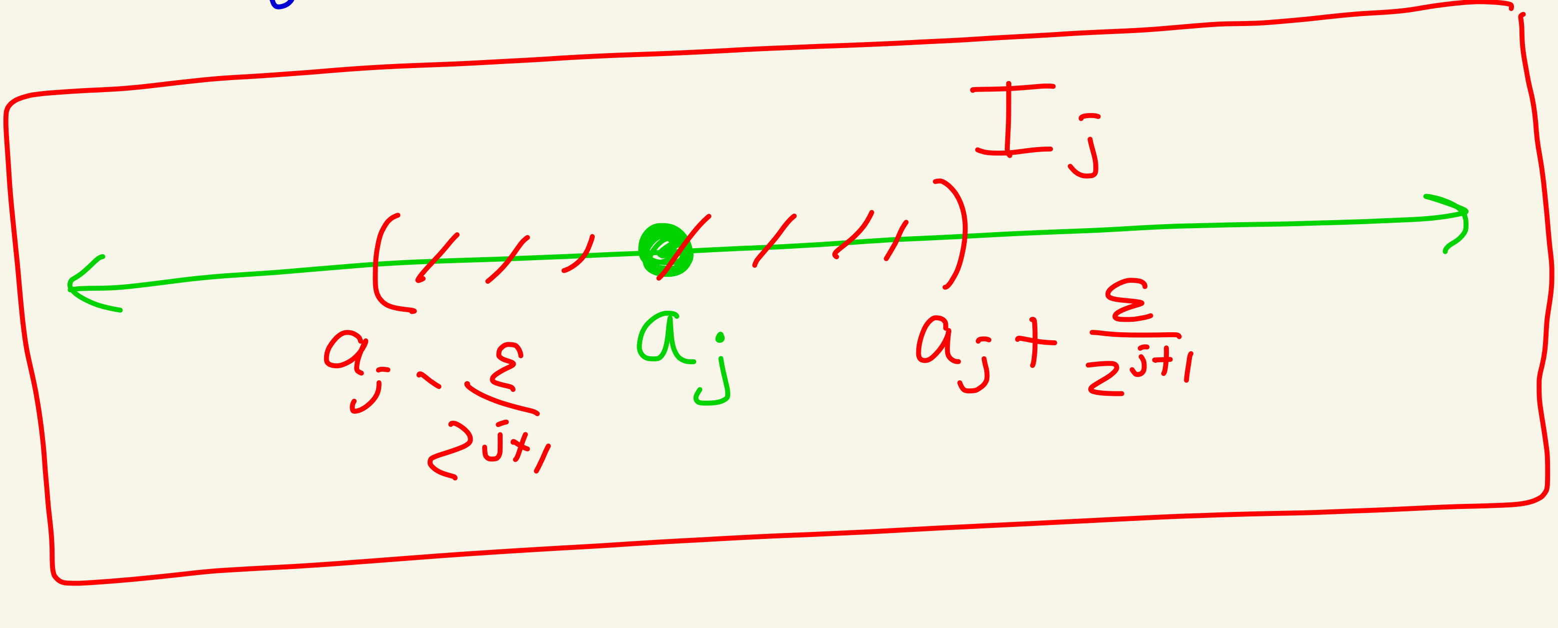
Thus, S has measure zero. $\blacksquare$

Theorem: Let $S \subseteq \mathbb{R}$. If S is countably infinite, then S has measure zero.

proof: Suppose $S = \{a_1, a_2, a_3, \ldots\}$

For each j, define

$$I_j = \left(a_j - \frac{\varepsilon}{2^{j+1}},\ a_j + \frac{\varepsilon}{2^{j+1}}\right)$$



We know $a_j \in I_j$.

Thus, $S \subseteq \bigcup_{j=1}^{\infty} I_j$.

Note,

$$\ell(I_j) = \left(a_j + \frac{\varepsilon}{2^{j+1}}\right) - \left(a_j - \frac{\varepsilon}{2^{j+1}}\right)$$

$$= 2 \cdot \frac{\varepsilon}{2^{j+1}} = \frac{\varepsilon}{2^j}.$$

Thus,

$$\sum_{j=1}^{\infty} \ell(I_j) = \sum_{j=1}^{\infty} \frac{\varepsilon}{2^j}$$

$$= \varepsilon \left[\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots \right]$$

last class

$$= \frac{\varepsilon}{2}\left[1 + \frac{1}{2} + \frac{1}{2^2} + \cdots\right] = \frac{\varepsilon}{2} \cdot 2$$

$= \varepsilon \leq \varepsilon$. Thus S has measure zero. $\blacksquare$

Ex: $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{Z}$ all have measure zero since they are countable.

Ex: The empty set has measure zero.

proof:

Let $\varepsilon > 0$.

Define $I_1 = \left(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right)$.

Then,

① $\phi \subseteq I_1$

$\boxed{\phi \subseteq X \text{ for any set } X}$

and ② $\ell(I_1) = \varepsilon$.

∎

Theorem: Let $A \subseteq \mathbb{R}$ and $B \subseteq \mathbb{R}$.
If $A \subseteq B$ and B has measure zero, then A has measure zero.

proof: In HW. $\blacksquare$

Theorem: [Prop 3, pg 19, Weir]
Let $A_1, A_2, A_3, \ldots$ be a countably infinite number of measure zero sets.
Then, $\bigcup_{k=1}^{\infty} A_k$ has measure zero.

Proof: On wednesday. $\blacksquare$