Math 5800

9/8/21

Def: Let $S \subseteq \mathbb{R}$.

We say that S is an <u>almost everywhere set</u> if

$$\mathbb{R} - S = \{ x \mid x \in \mathbb{R} \text{ and } x \notin S \}$$

has measure zero

Ex: Let S be the set of irrational numbers.

Then, $\mathbb{R} - S = \mathbb{Q}$ which has measure zero because $\mathbb{Q}$ is countable.

Thus, S is an almost everywhere set.

Theorem: If $S_1, S_2, \ldots, S_n$ are almost everywhere sets, then $\bigcap_{k=1}^{n} S_k$ is an almost everywhere set.

For proofs use:
$$\mathbb{R} - \bigcap_k S_k = \bigcup_k (\mathbb{R} - S_k)$$

pf: HW $\blacksquare$

Theorem: Let $S_1, S_2, S_3, \ldots$ be a countably infinite number of almost everywhere sets. Then, $\bigcap_{k=1}^{\infty} S_k$ is an almost everywhere set.

pf: HW $\blacksquare$

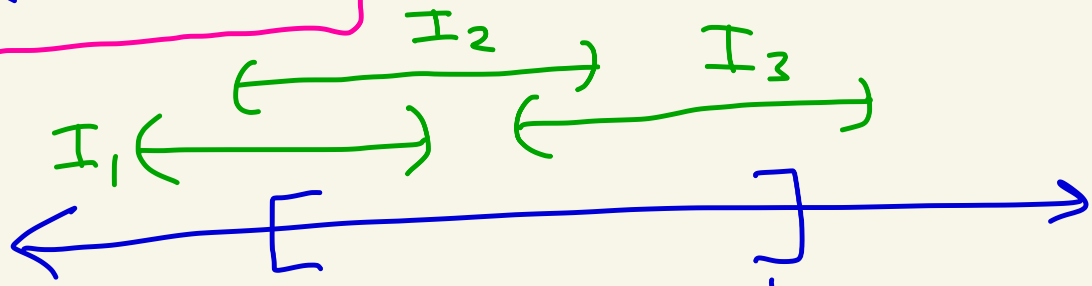
We will now give an example of a set that doesn't have measure zero. But first a lemma.

<u>Lemma:</u> Let $a, b \in \mathbb{R}$ with $a < b$. If $I_1, I_2, \ldots, I_n$ are n bounded open intervals where

$$[a,b] \subseteq \bigcup_{k=1}^{n} I_k$$

then

$$\sum_{k=1}^{n} \ell(I_k) > b-a$$



n=3 picture

proof by induction:

Let $S(n)$ be the statement of the lemma.

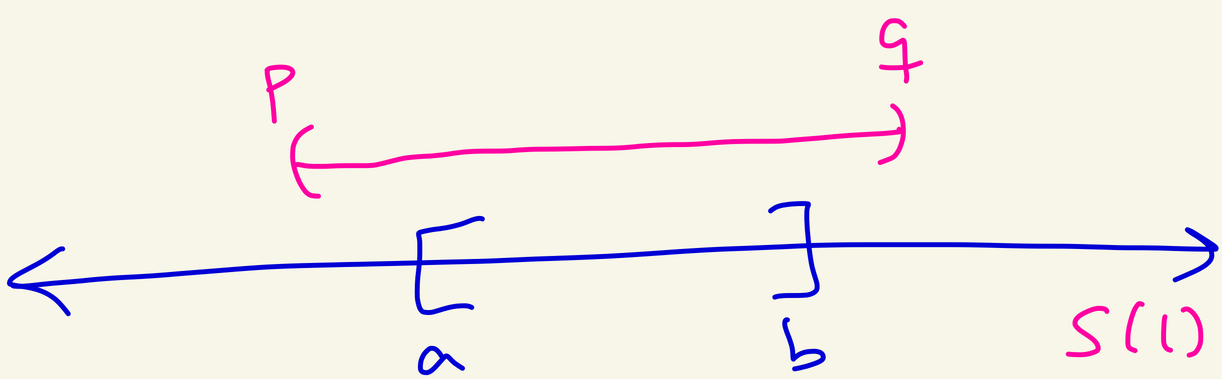
base case $S(1)$: Suppose $a, b \in \mathbb{R}$ with $a < b$. Suppose I_1 is a bounded open interval with $[a,b] \subseteq I_1$.

Let $I_1 = (p, q)$.

Then, $p < a < b < q$.

And, $\ell(I_1) = q - p > b - a$

$q > b$
$-p > -a$



S(1) picture

<u>Induction step:</u> Let $n \geq 1$ and suppose $S(n)$ is true.

We now prove $S(n+1)$ is true.

Let $a, b \in \mathbb{R}$ with $a < b$.

Suppose $[a,b] \subseteq \bigcup_{k=1}^{n+1} I_k$

where $I_1, I_2, \ldots, I_n, I_{n+1}$ are bounded open intervals.

Since $[a,b] \subseteq I_1 \cup I_2 \cup \cdots \cup I_n \cup I_{n+1}$.

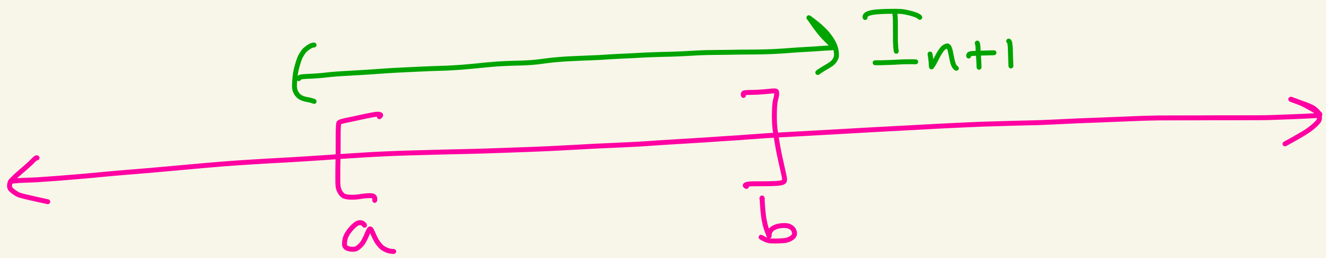
we must have that $b \in I_k$ for some k.

By relabeling the intervals we can assume that $b \in I_{n+1}$.

Suppose $I_{n+1} = (c, d)$.

Since $b \in I_{n+1}$, we know $c < b < d$.

<u>Case 1: Suppose $c \leq a$.</u>



Then,

$$\ell(I_{n+1}) = d - c \geq b - a$$

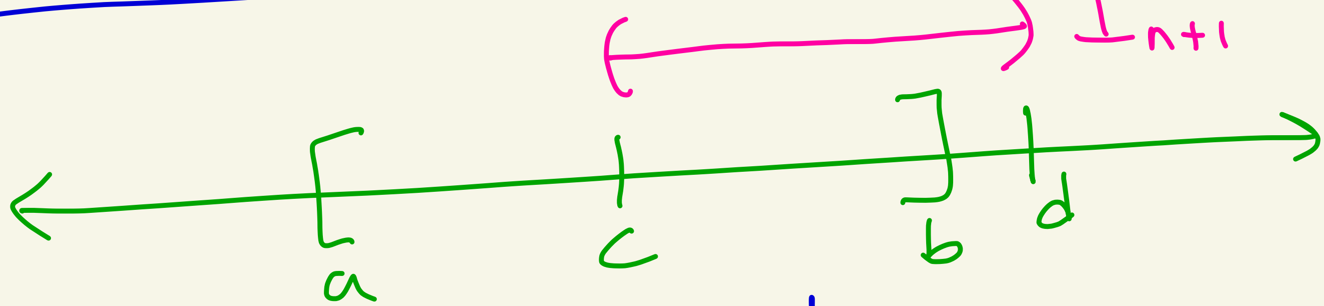
$$\boxed{\begin{array}{c} d > b \\ -c \geq -a \end{array}}$$

Thus,

$$\sum_{k=1}^{n+1} \ell(I_k) = \underbrace{\left[\sum_{k=1}^{n} \ell(I_k)\right]}_{>0} + \underbrace{\ell(I_{n+1})}_{\geq b-a}$$

$$> b - a$$

Case 2: Suppose $a < c$



Then, $a < c < b < d$.

And, $[a, c] \subseteq \bigcup_{k=1}^{n} I_k \quad (*)$

By the induction hypothesis applied to $(*)$ we get that

$$\sum_{k=1}^{n} \ell(I_k) > c - a.$$

Thus,

$$\sum_{k=1}^{n+1} \ell(I_k) = \left[\sum_{k=1}^{n} \ell(I_k)\right] + \ell(I_{n+1})$$
$$> (c-a) + (d-c)$$
$$= d - a > b - a.$$

By case 1 and case 2, $S(n+1)$ is true when $S(n)$ is true. By induction we are done. $\blacksquare$

Theorem: Let $a, b \in \mathbb{R}$ with $a < b$. Then $[a,b]$ does not have measure zero.

proof: Suppose

$I_1, I_2, I_3, \ldots$

is a sequence of bounded open intervals with

$$[a,b] \subseteq \bigcup_{k=1}^{\infty} I_k.$$

open cover of $[a,b]$

$[a,b]$ is closed and bounded

4650: (Heine-Borel)
Every open cover of a closed, bounded subset of $\mathbb{R}$ has a finite sub-cover

By Heine-Borel, there must be $n \geq 1$ where

$$[a,b] \subseteq \bigcup_{k=1}^{n} I_k$$

By the lemma, $\sum_{k=1}^{n} \ell(I_k) > b-a$.

Thus, $\sum_{k=1}^{\infty} \ell(I_k) \geq \sum_{k=1}^{n} \ell(I_k) > b-a$.

Therefore, if say you pick $\varepsilon = b-a$ there is no sequence of bounded open intervals $I_1, I_2, \ldots$ with $[a,b] \subseteq \bigcup_{k=1}^{\infty} I_k$ and $\sum_{k=1}^{\infty} \ell(I_k) < \varepsilon$

[because $\sum_{k=1}^{\infty} \ell(I_k) > b-a$]

Thus, $[a,b]$ does not have measure zero. ∎

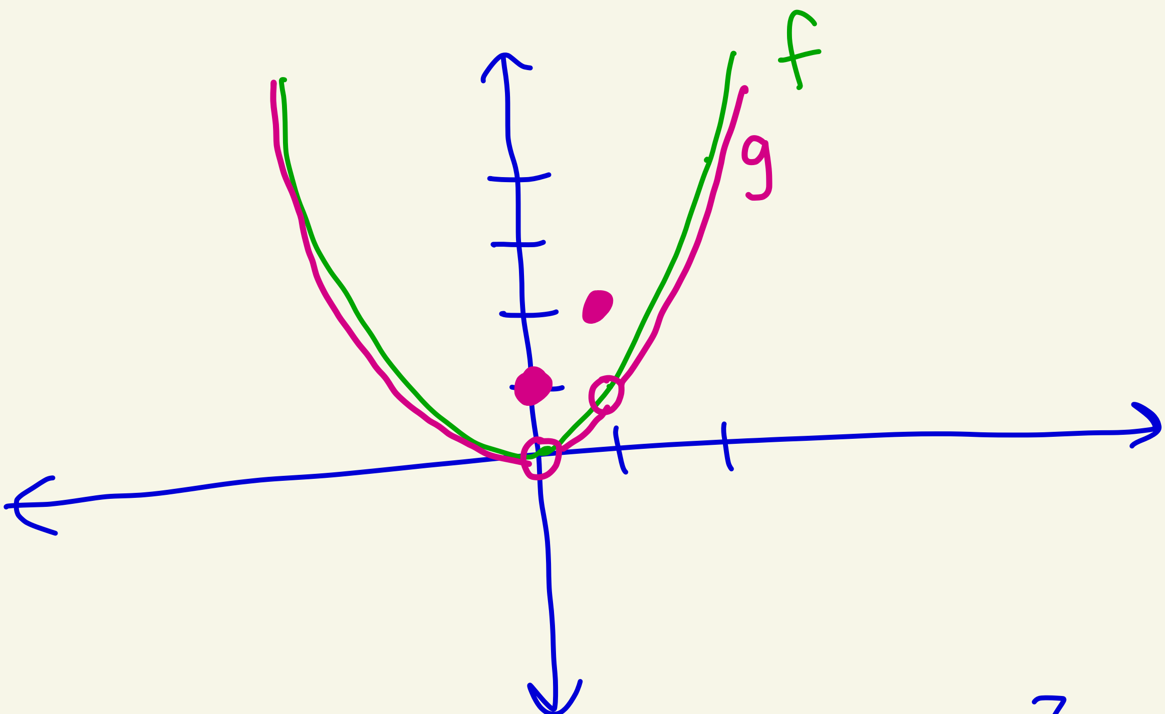
Def: Let $A \subseteq \mathbb{R}$ and $f: A \to \mathbb{R}$ and $g: A \to \mathbb{R}$. We say that <u>$f = g$ almost everywhere on A</u>, or $f = g$ a.e. on A, if the set
$$\{x \in A \mid f(x) \neq g(x)\}$$
has measure zero.

Note: If $A = \mathbb{R}$ and $f = g$ almost everywhere on $\mathbb{R}$, then we just say $f = g$ almost everywhere

Ex: Let $A = \mathbb{R}$,

$f(x) = x^2$ for all $x \in \mathbb{R}$,

and $g(x) = \begin{cases} 1 & \text{if } x = 0 \\ 2 & \text{if } x = 1 \\ x^2 & \text{if } x \neq 0 \text{ and } x \neq 1 \end{cases}$



Then, $\{x \in A \mid f(x) \neq g(x)\}$
$= \{0, 1\}$ which has measure zero

So, $f = g$ almost everywhere on $\mathbb{R}$.

Ex: Let $A = [0,1]$.

Let $f : [0,1] \to \mathbb{R}$ and
$g : [0,1] \to \mathbb{R}$
be defined as follows:

$g(x) = 1$ for all $x \in [0,1]$

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \cap [0,1] \\ 1 & \text{otherwise} \end{cases}$$

Then,

$$\{x \in [0,1] \mid f(x) \neq g(x)\}$$
$$= \mathbb{Q} \cap [0,1].$$

Since $\mathbb{Q} \cap [0,1] \subseteq \mathbb{Q}$ and $\mathbb{Q}$ has measure zero, we know $\mathbb{Q} \cap [0,1]$ has measure zero. Thus, $f = g$ almost everywhere on $[0,1]$.