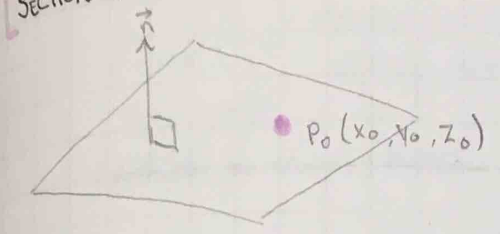
SECTION 12.1 - PLANES & SURFACES

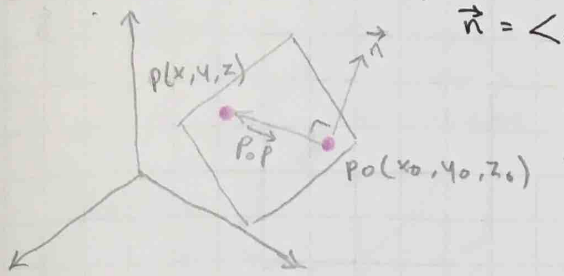


- A plane is determined by a point $P_0(x_0, y_0, z_0)$ in the plane and a vector $\vec{n}$ that is orthogonal / perpendicular to the plane
- $\vec{n}$ is called **the normal vector** of the plane

~ DERIVATION OF PLANE EQUATION ~

Let a plane be determined by a point $P_0(x_0, y_0, z_0)$ on the plane and a normal vector

$$\vec{n} = \langle a, b, c \rangle$$



Let $p(x,y,z)$ be any other point on the plane

Then $\left[\vec{P_0P} = \langle x - x_0, y - y_0, z - z_0 \rangle\right]$ lies in the plane.

Then $\vec{n}$ and $\vec{P_0P}$ are perpendicular.

Then $\vec{n} \cdot \vec{P_0P} = 0$ ← DOT PRODUCT

So, $\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$

$$\boxed{a(x - x_0) + b(y - y_0) + c(z - z_0) = 0}$$ EQUATION OF A PLANE

$$ax + by + cz = ax_0 + by_0 + cz_0$$

$$\boxed{ax + by + cz = d \quad \leftarrow (d = ax_0 + by_0 + cz_0)}$$

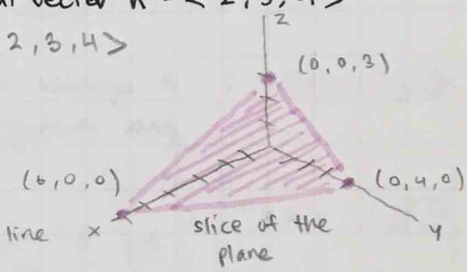
EX: Find an equation of the plane through the point $(2, 4, -1)$ with normal vector $\vec{n} = \langle 2, 3, 4 \rangle$

$(x_0, y_0, z_0) = (2, 4, -1)$ $\quad \langle a, b, c \rangle = \langle 2, 3, 4 \rangle$

$$2(x-2) + 3(y-4) + 4(z+1) = 0$$

$$2x + 3y + 4z = 4 + 12 - 4$$

$$\boxed{2x + 3y + 4z = 12}$$

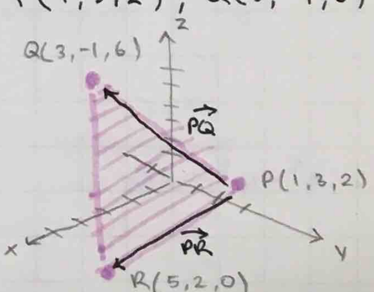


not on the same line
↓

EX: Find an equation of the plane that passes through the non-colinear points $P(1,3,2)$, $Q(3,-1,6)$ and $R(5,2,0)$

Consider $\vec{PQ} = \langle 3-1, -1-3, 6-2 \rangle = \langle 2, -4, 4 \rangle$

and $\vec{PR} = \langle 5-1, 2-3, 0-2 \rangle = \langle 4, -1, -2 \rangle$



Then $\vec{PQ}$ and $\vec{PR}$ lie on the plane. So $\vec{n} = \vec{PQ} \times \vec{PR}$ is perpendicular to $\vec{PQ}$ and $\vec{PR}$, so $\vec{n}$ is a normal vector for the plane.

$$\vec{n} = \vec{PQ} \times \vec{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -4 & 4 \\ 4 & -1 & -2 \end{vmatrix} = \hat{i}(8+4) - \hat{j}(-4-16) + \hat{k}(-2+16) = 12\hat{i} + 20\hat{j} + 14\hat{k} = \langle 12, 20, 14 \rangle$$

$\vec{n} = \langle 12, 20, 14 \rangle$

$P_0 = (1, 3, 2)$

$$12(x-1) + 20(y-3) + 14(z-2) = 0$$

$$\boxed{12x + 20y + 14z = 100}$$

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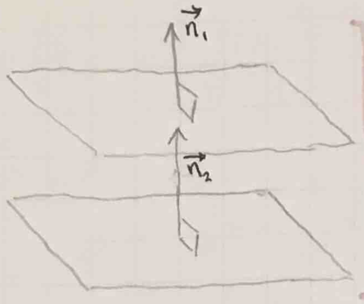
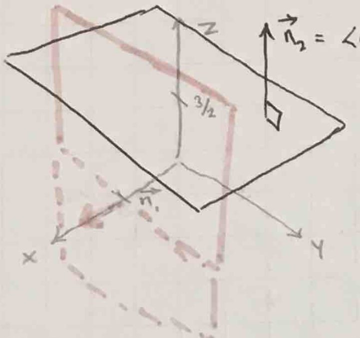


Two planes are parallel if their normal vectors are parallel, i.e. one of the normal vectors is a multiple of the other.

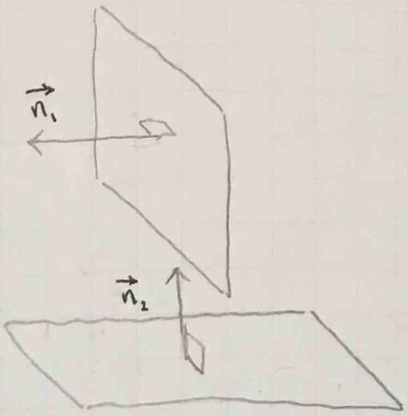
i.e. $\vec{n}_1 = c\vec{n}_2$, c is a constant (could be negative)



$\vec{n}_2 = \langle 0,0,2 \rangle$

Two planes are perpendicular if their normal vectors are, that is

$$\vec{n}_1 \cdot \vec{n}_2 = 0$$



EX: $2x - 3y + 4z = 1$ has normal vector $\vec{n}_1 = \langle 2, -3, 4 \rangle$

Let $\vec{n}_2 = 2\vec{n}_1 = \langle 4, -6, 8 \rangle$, then $4x - 6y + 8z = 17$

↑ **parallel** because scalar multiple of $2x - 3y + 4z = 1$

EX: Is $x = 1$ perpendicular to $2z = 3$

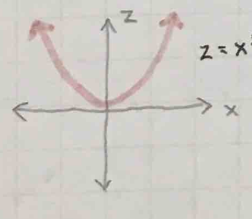
$x = 1 \longleftarrow x + 0y + 0z = 1 \longleftarrow \vec{n}_1 = \langle 1, 0, 0 \rangle$

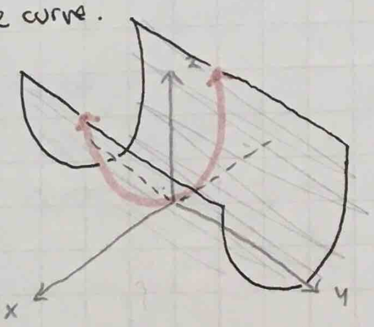
$2z = 3 \longleftarrow 0x + 0y + 2z = 3 \longleftarrow \vec{n}_2 = \langle 0, 0, 2 \rangle$

$\vec{n}_1 \cdot \vec{n}_2 = 1 \cdot 0 + 0 \cdot 0 + 0 \cdot 2 = 0 \rightarrow$ **Yes**, they are perpendicular.

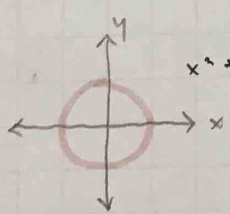
cylinder: A cylinder is a surface that consists of all lines that are parallel to a given line and pass through a given ~~space~~ plane curve.

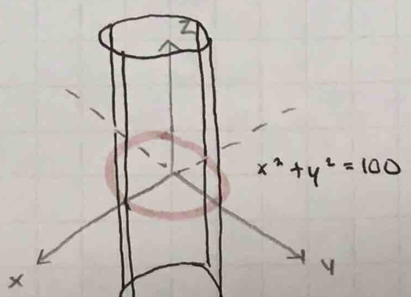
EX: $z = x^2$



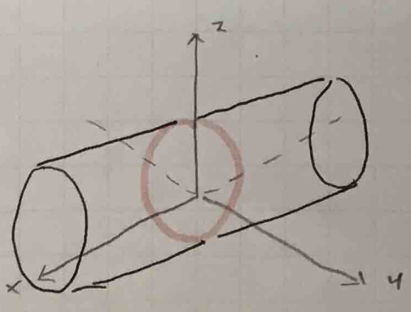


EX: $x^2 + y^2 = 100$





EX: $y^2 + z^2 = 1$

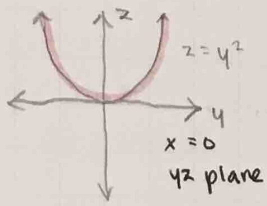


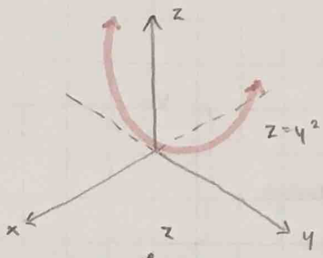
Quadratic Surfaces:

EX: $z = 4x^2 + y^2$

<u>$x=0$ trace</u>

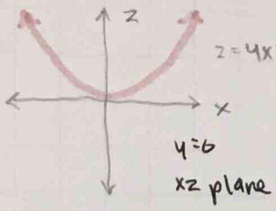
$z = y^2$

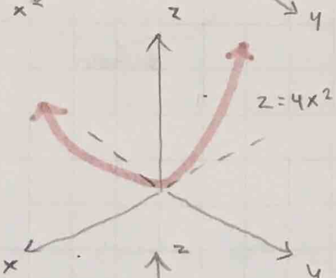




<u>$y=0$ trace</u>

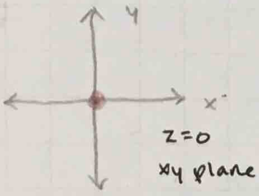
$z = 4x^2$

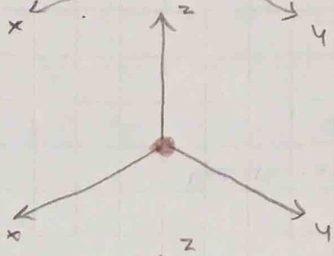




<u>$z=0$ trace</u>

$0 = 4x^2 + y^2$

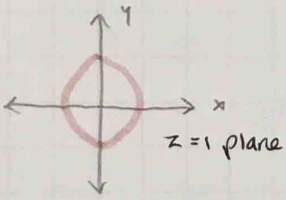


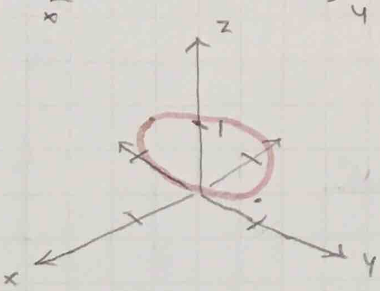


<u>$z=1$ trace</u>

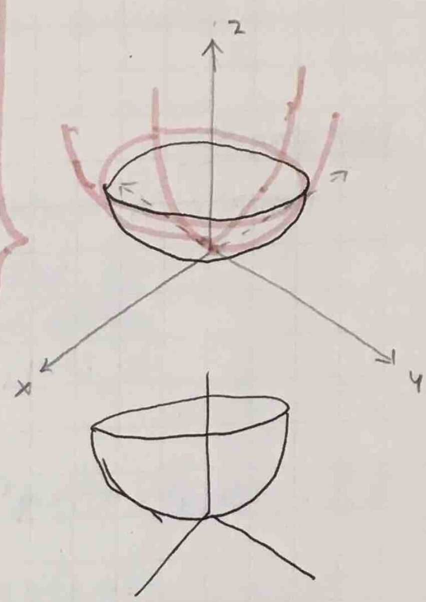
$1 = 4x^2 + y^2$

$1 = \frac{x^2}{(\frac{1}{2})^2} + \frac{y^2}{1^2}$

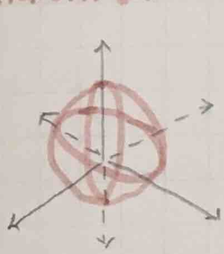




Elliptic Paraboloid

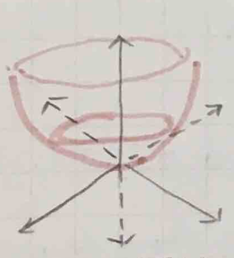


~ Classification of Quadratic Surfaces ~



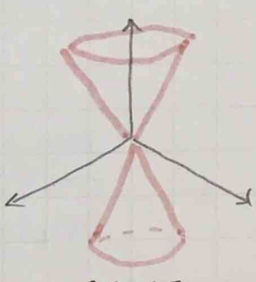
ELLIPSOID

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$



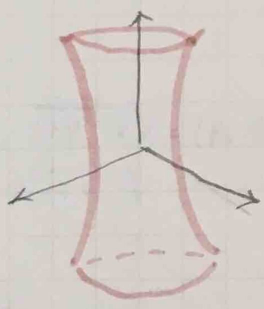
ELLIPTIC PARABOLOID

$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$



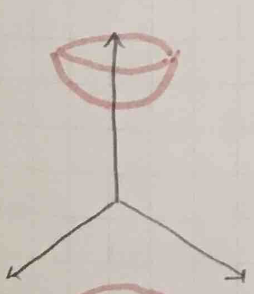
CONE

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$$



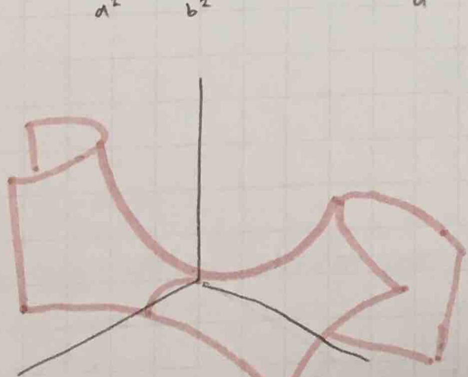
HYPEROLOID OF ONE SHEET

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$



HYPEROLOID OF TWO SHEETS

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$



HYPERBOLIC PARABOLOID

(SADDLE)

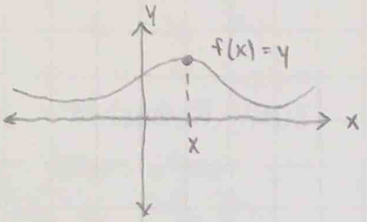
$$z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

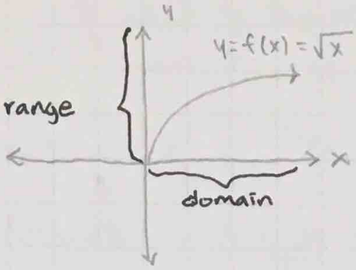
[Section 12.2 - Graphs and Level "Curves"]

27 August 2019

In 2D

$y = f(x)$

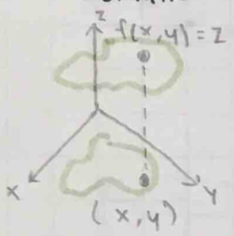




- Domain is x-values that make sense to plug into $f(x) = \sqrt{x}$
  Domain $= [0, \infty)$
- Range = possible outputs of $f(x)$
  Range $= [0, \infty)$

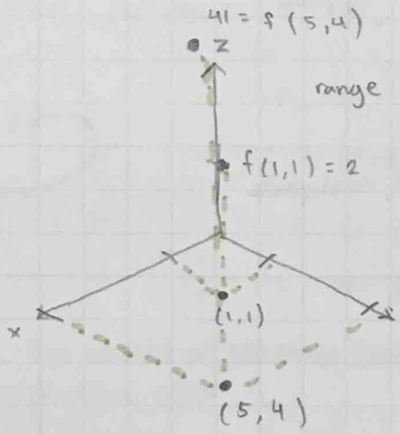
What about 3D?

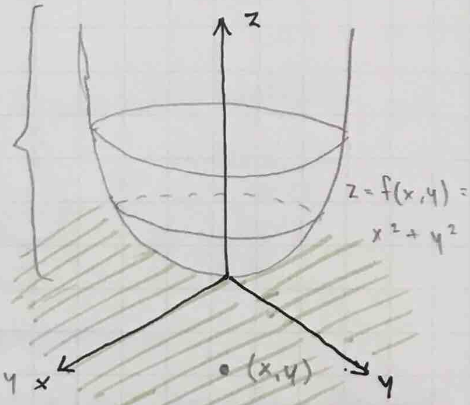
Definition: A function f of two variables is a rule that assigns to each ordered pair of real numbers (x, y) in a set D, called the domain of f, a unique real number denoted by $f(x,y)$. The range of f is the possible $f(x,y)$ values as (x,y) varies through D.



EX: $f(x,y) = x^2 + y^2$

(x,y)	$z = f(x,y) = x^2 + y^2$
$(1,1)$	$1^2 + 1^2 = 2$
$(2,-2)$	$2^2 + (-2)^2 = 8$
$(4,5)$	$4^2 + 5^2 = 41$





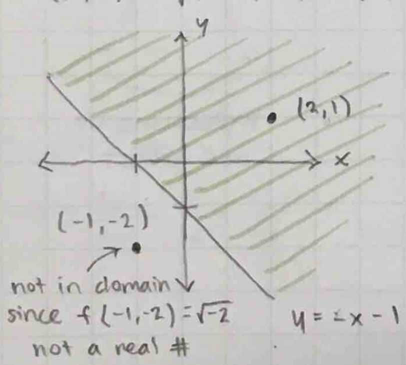
D = domain is all (x,y)
R = possible z-values

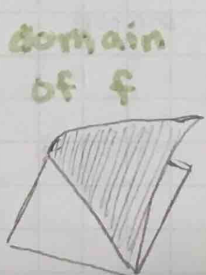
EX: $f(x,y) = \sqrt{x+y+1}$

$f(2,1) = \sqrt{2+1+1} = \sqrt{4} = 2$

What is the domain of f? I.e. what (x,y) can we plug into f?

To plug (x,y) into f we need $x + y + 1 \geq 0$
$y \geq -x - 1$



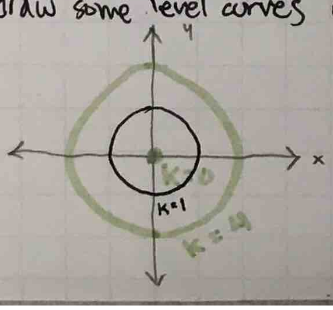


Definition: The level curves of a function f of two variables are the curves with equations $k = f(x,y)$ where k is any constant in the range of f.

EX: Find and draw some level curves of $f(x,y) = x^2 + y^2$.

$k=0$
$0 = x^2 + y^2$

$k=1$
$1 = x^2 + y^2$



$k=4$
$4 = x^2 + y^2$
$2^2 = x^2 + y^2$

In general, above the level curve $k = f(x,y)$ the z-height is k.