Tuesday
9/10

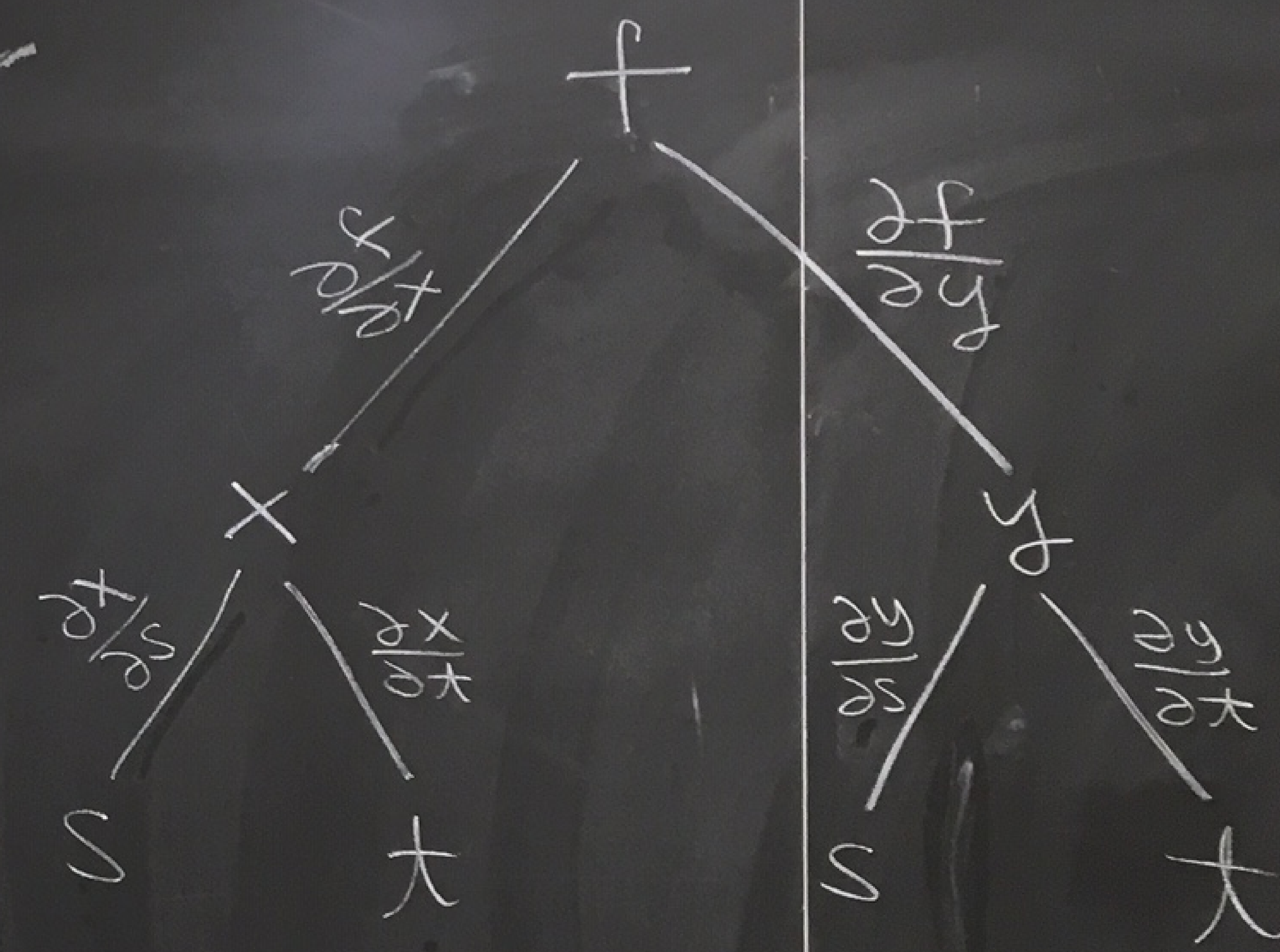
Ex from last time

$f(x,y) = x^2 y$

$x = s^2 + t^2$

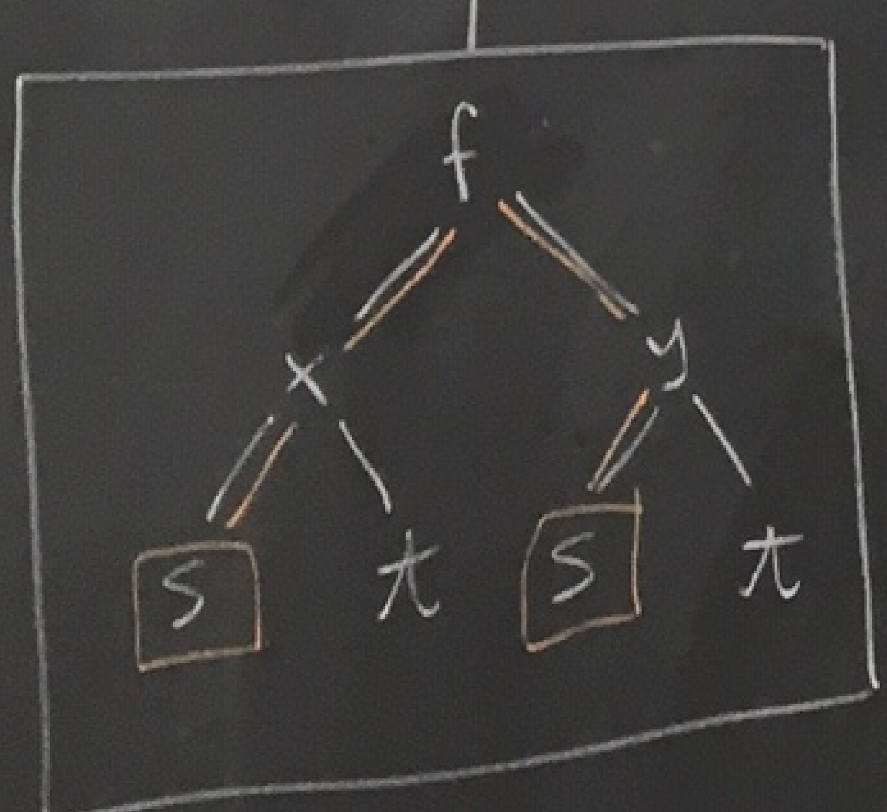
$y = 2st$

Find $\frac{\partial f}{\partial s}$ and $\frac{\partial f}{\partial t}$.



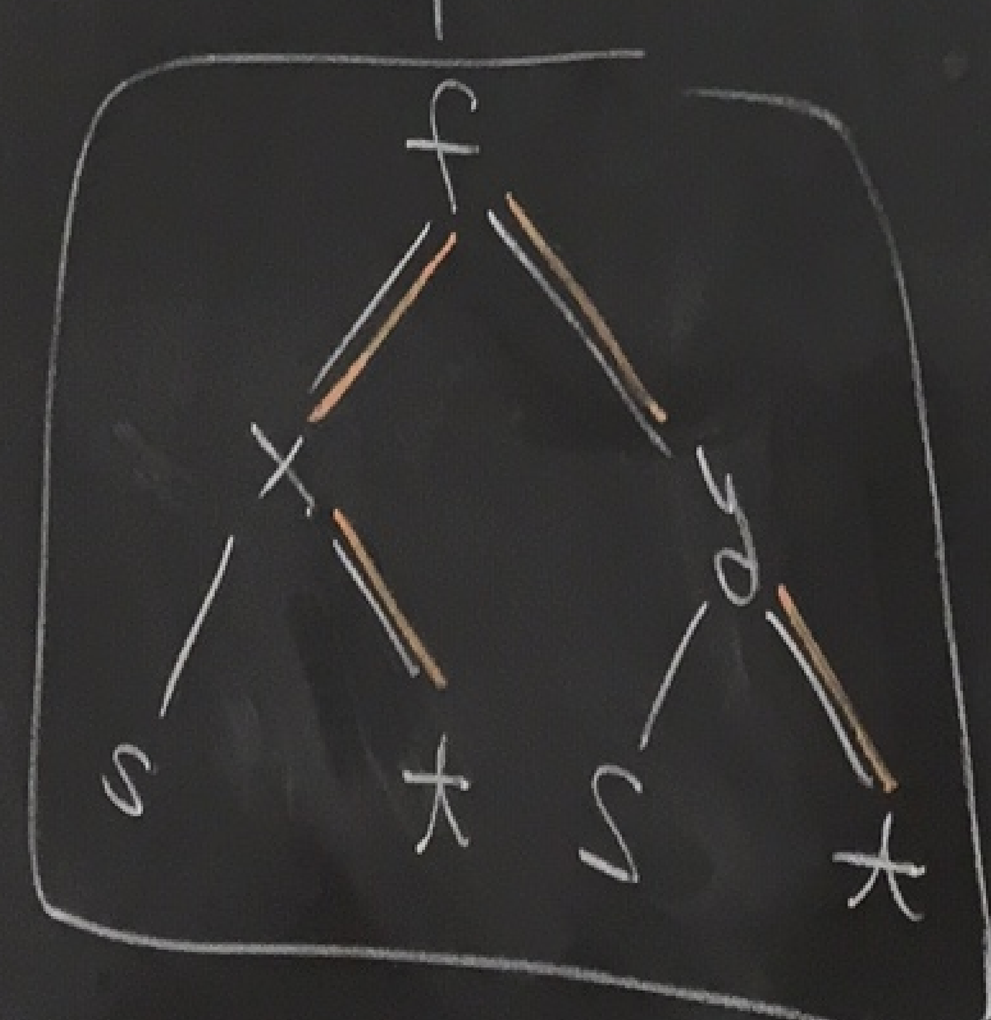
$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} = \left(y(2x)\right)(2s) + (x^2)(2t)$$

$$= 4(\underbrace{s^2+t^2}_{x})(\underbrace{2st}_{y})(s) + (\underbrace{s^2+t^2}_{x^2})^2(2t)$$



$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} = (2xy)(2t) + (x^2)(2s)$$

$$= 4(\underbrace{s^2+t^2}_{x})(\underbrace{2st}_{y})t + (\underbrace{s^2+t^2}_{x^2})^2(2s)$$



Ex: If $u = x^4 y + y^2 z^3$

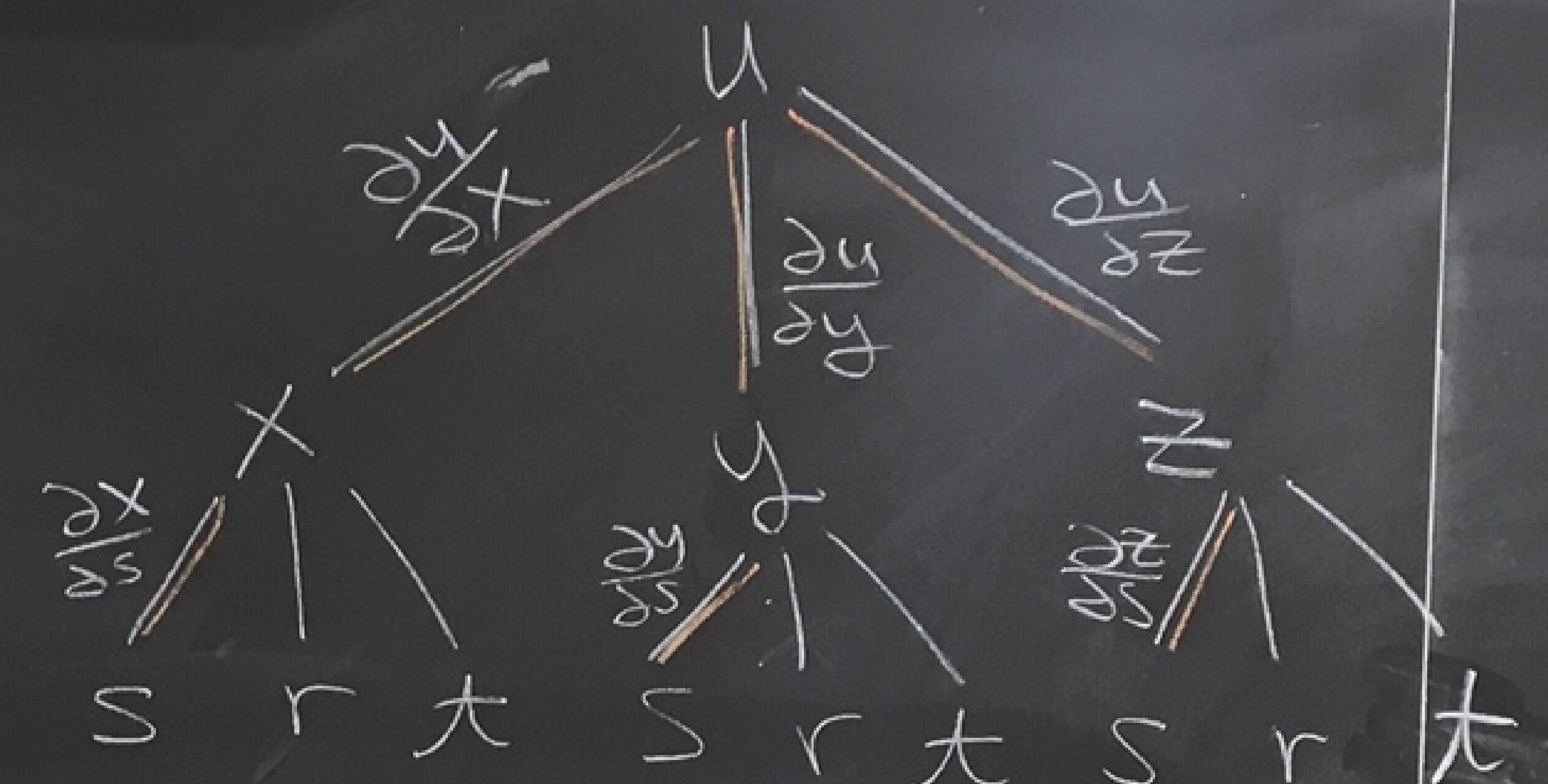
where $x = rse^{t}$

$y = rs^2 e^{-t}$

and $z = sr^2 \sin(t)$.

Find $\dfrac{\partial u}{\partial s}$ where

$r = 2, s = 1$, and $t = 0$.



$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial s}$$

$$= (4x^3y)(re^{t}) + (x^4 + 2yz^3)(2rse^{-t})$$

$$+ (3y^2z^2)(r^2\sin(t))$$

when $r = 2$, s

we have

$x = (2)$

$y = (2$

$z = (1$

$e^{-t})$

$(t)) =$

$$= (4(2)^3(2))(2e^0) + (2^4 + 2(2)(0)^3)(2(2)(1)e^{-0})$$
$$+ (3(2)^2(0)^2)(2^2\sin(0))$$
$$= 192.$$

Plug these in

When
$r = 2, s = 1, t = 0$
we have

$x = (2)(1)e^0 = 2$

$y = (2)(1)^2e^{-0} = 2$

$z = (1)(2)^2\sin(0) = 0$

12.6 — Directional Derivatives and the Gradient

Given $z = f(x,y)$, the partial derivatives $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ represent the rate of change of $z = f(x,y)$ at (x_0, y_0) in the x-direction and the y-direction.

What about other directions?

unit vector means length 1

De
of
in
uni

nt

atives

he

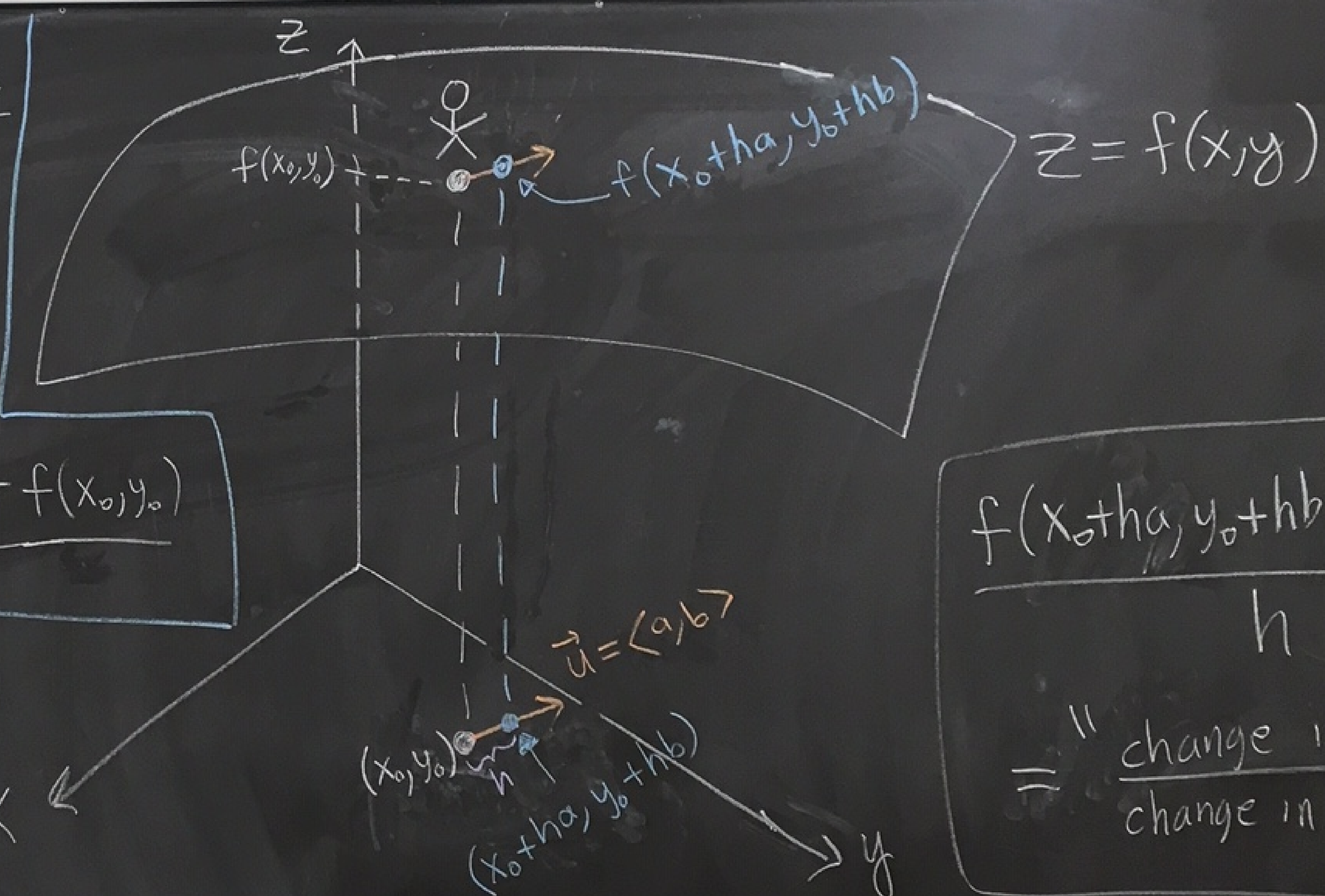
e

Def: The directional derivative of $f(x,y)$ at (x_0,y_0) in the direction of a unit vector $\vec{u}=\langle a,b\rangle$ is

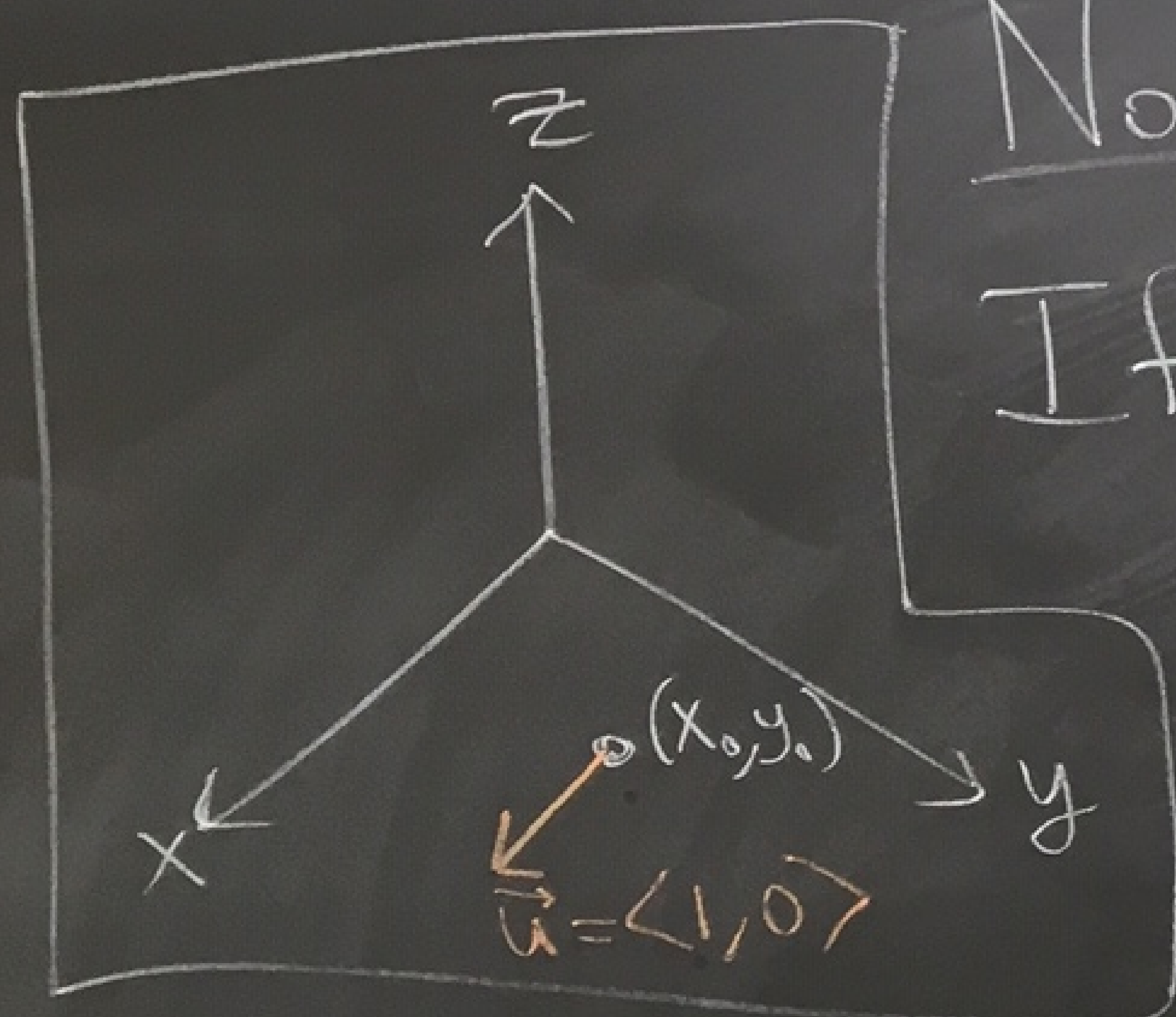
$$D_{\vec{u}} f(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0+ha,\,y_0+hb)-f(x_0,y_0)}{h}$$

if the limit exists.

unit vector means length 1



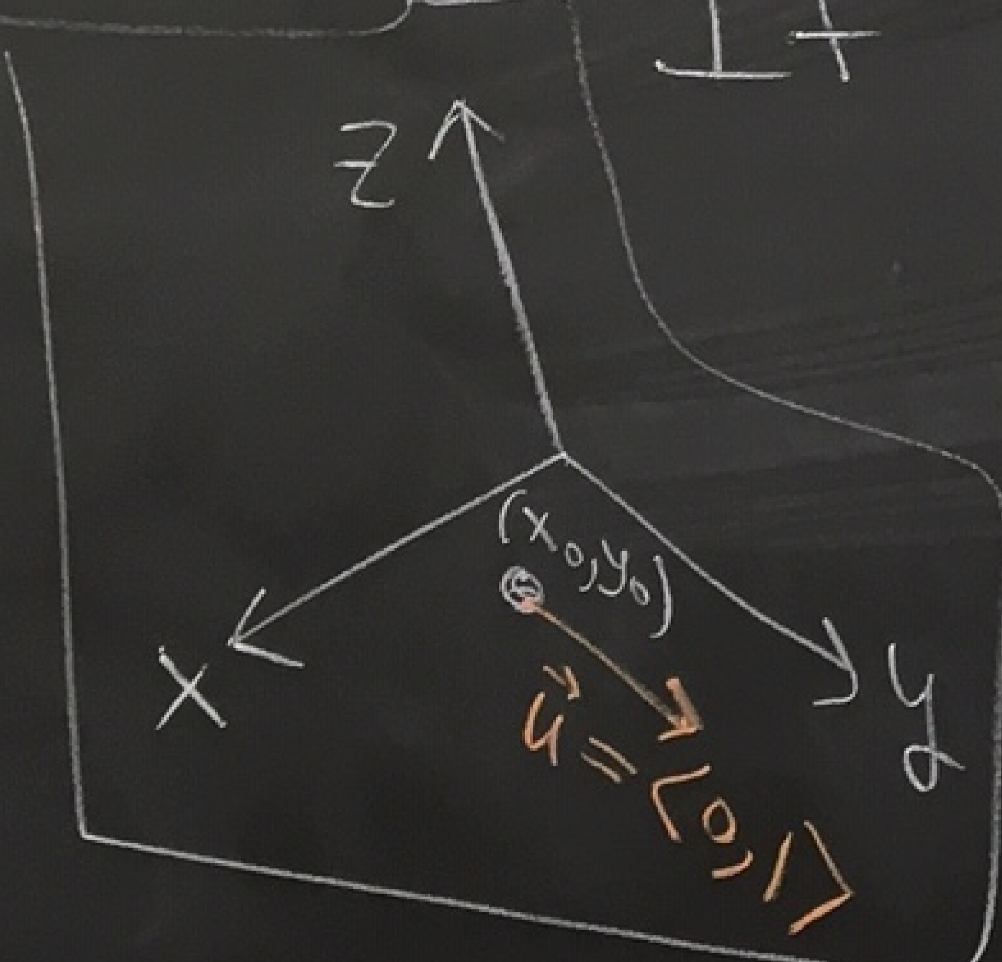
$$\boxed{\frac{f(x_0+ha,\,y_0+hb)-f(x_0,y_0)}{h} = \text{“}\frac{\text{change in } z}{\text{change in } x,y}\text{”}}$$



Note:

If $\vec{u}=\langle 1,0\rangle$, then

$$D_{\vec{u}} f(x_0,y_0)=\lim_{h\to 0}\frac{f(x_0+h,y_0)-f(x_0,y_0)}{h}=f_x(x_0,y_0)$$



If $\vec{u}=\langle 0,1\rangle$, then

$$D_{\vec{u}} f(x_0,y_0)=\lim_{h\to 0}\frac{f(x_0,y_0+h)-f(x_0,y_0)}{h}=f_y(x_0,y_0)$$

Theorem: If $f(x,y)$ is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\vec{u} = \langle a, b \rangle$ and

$$D_{\vec{u}} f(x_0, y_0) = f_x(x_0,y_0) \cdot a + f_y(x_0,y_0) \cdot b$$

$$= \underbrace{\langle f_x(x_0,y_0), f_y(x_0,y_0) \rangle}_{\text{this is called the gradient vector of } f \text{ at } (x_0,y_0) \text{ and is denoted by } \nabla f(x_0,y_0)} \cdot \langle a, b \rangle$$

$$= \nabla f(x_0,y_0) \cdot \vec{u}$$

Ex: Find the directional derivative of $f(x,y) = x^3 - 3xy + 4y^2$ at $(1,2)$ in the direction of $\vec{u} = \langle \frac{\sqrt{3}}{2}, \frac{1}{2} \rangle$.

Note: $|\vec{u}| = \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2}$

$= \sqrt{\frac{3}{4} + \frac{1}{4}} = 1$

So, $\vec{u}$ is a unit vector. ✓

f

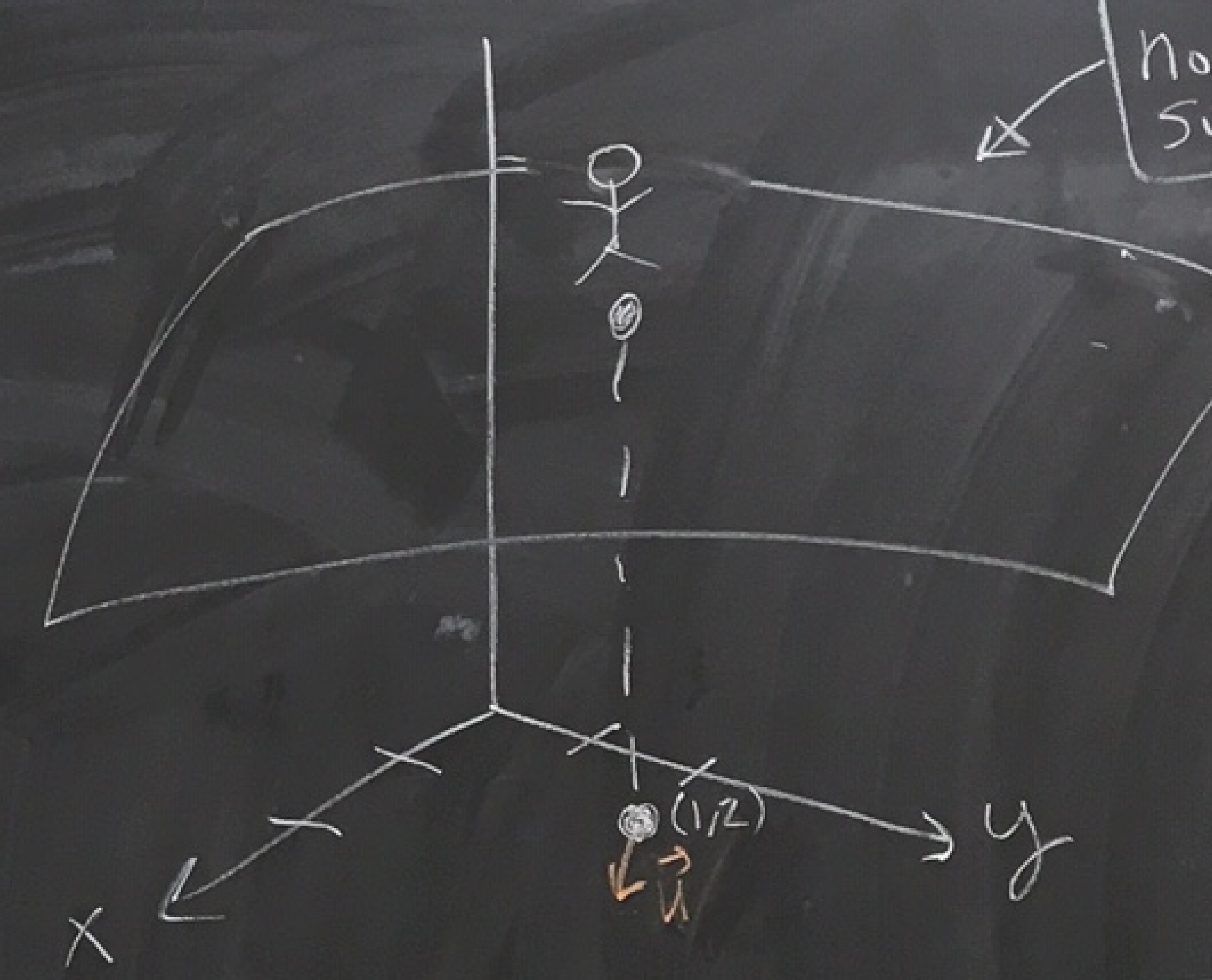
f

$\nabla f($

$D_{\vec{u}} f$

$$f_x = 3x^2 - 3y$$
$$f_y = -3x + 8y$$

$$\nabla f(1,2) = \langle f_x(1,2), f_y(1,2) \rangle$$
$$= \langle 3(1)^2 - 3(2), -3(1) + 8(2) \rangle$$
$$= \langle -3, 13 \rangle$$



$z = x^3 - 3xy + 4y^2$

The slope if you are on the surface above (1,2) in the direction of $\vec{u}$ is ≈ 3.9

$$D_{\vec{u}} f(1,2) = \nabla f(1,2) \cdot \vec{u}$$
$$= \langle -3, 13 \rangle \cdot \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2} \right\rangle = (-3)\left(\frac{\sqrt{3}}{2}\right) + (13)\left(\frac{1}{2}\right) = \frac{13 - 3\sqrt{3}}{2} \approx 3.901923788646684\ldots$$