9/19
Thursday

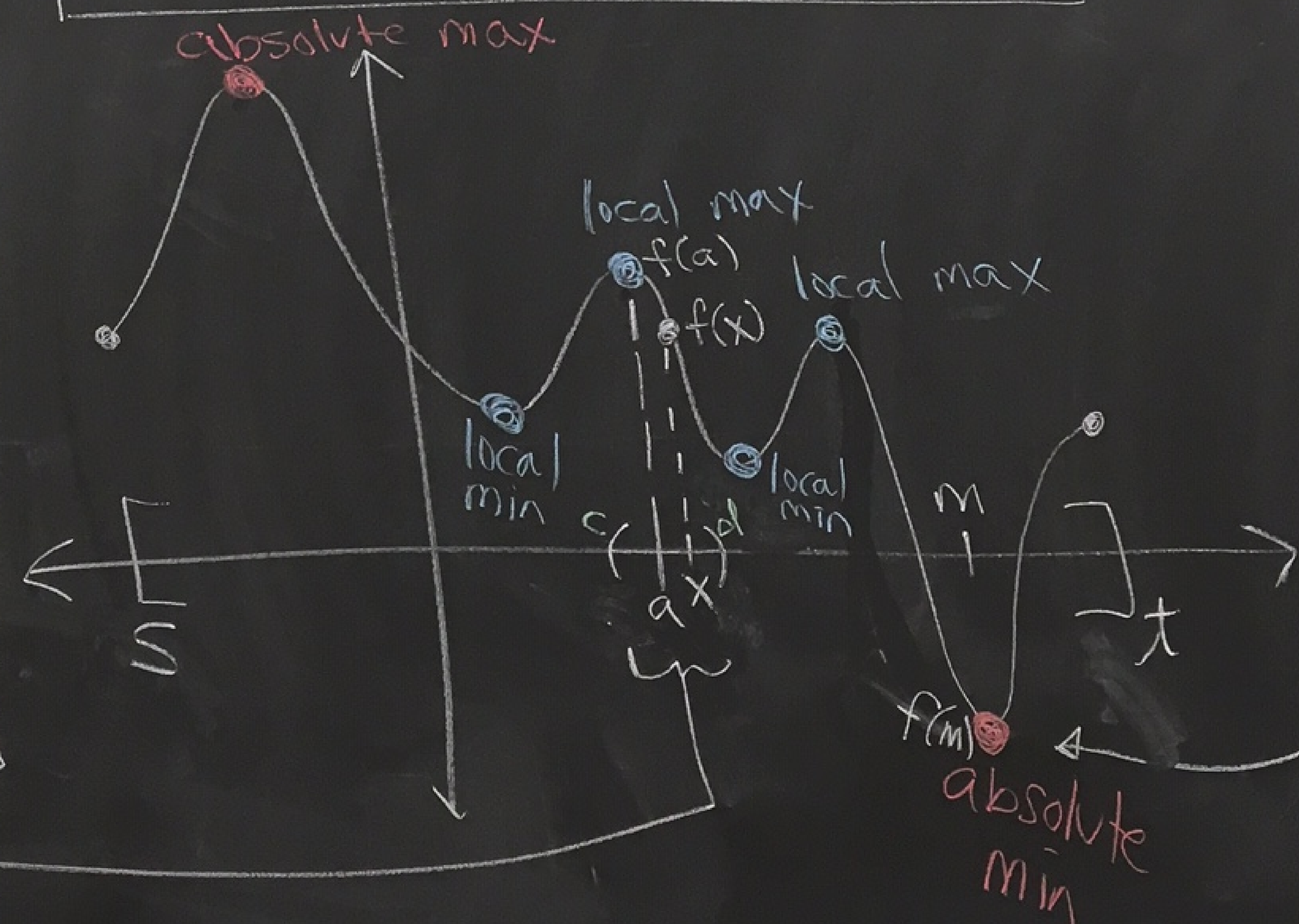
12.8 - Max & Min Problems

Recal Calc I

a is a local max. of f if there is an interval (c,d) around a such that

$$f(x) \leq f(a)$$

for all x in (c,d)



m is an absolute min. of f on [s,t] if

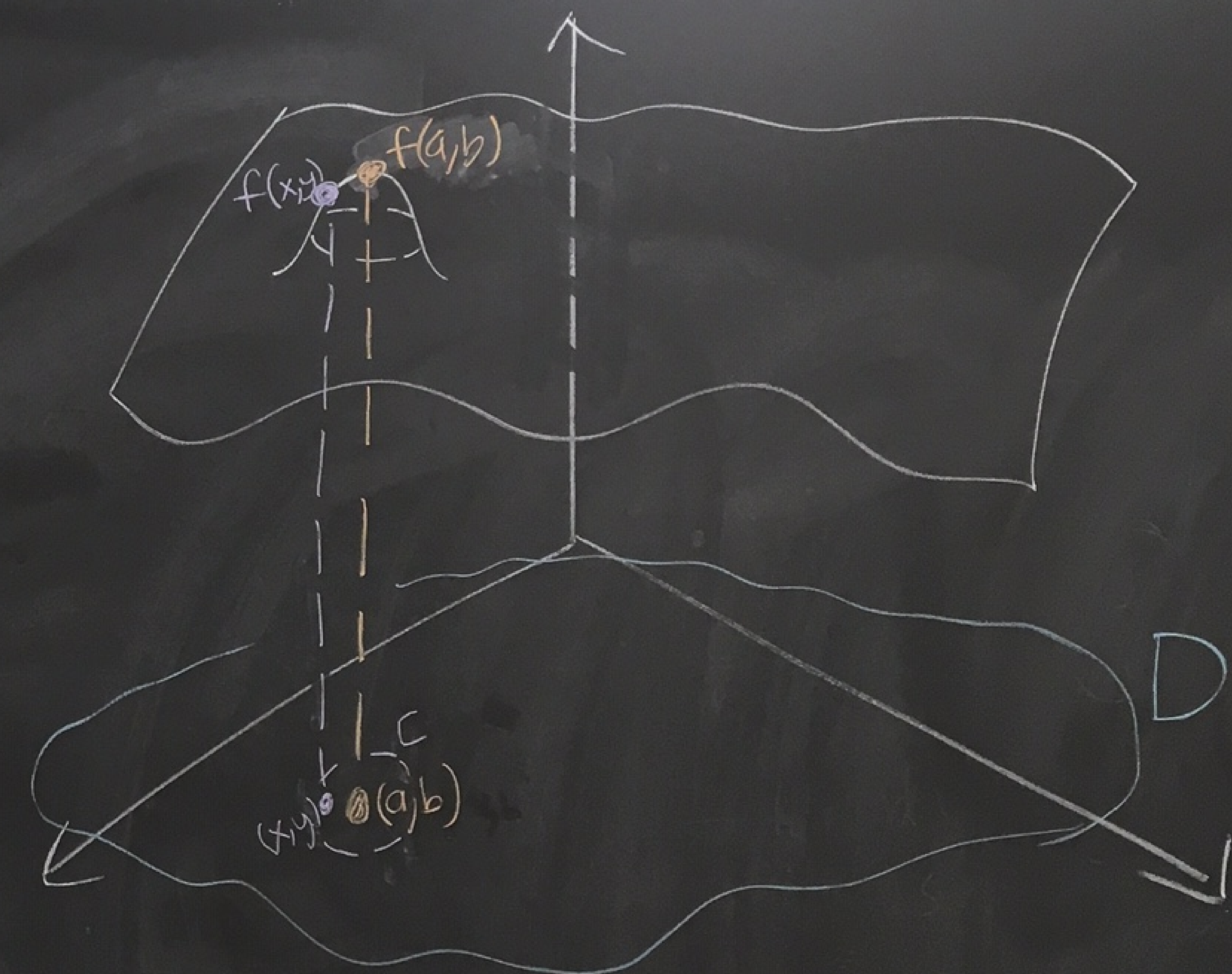
$$f(m) \leq f(x)$$

for all x in [s,t]

Def: Let $f(x,y)$ be a function of two variables with domain D.

- (a,b) is a <u>local maximum</u> of f if $f(x,y) \le f(a,b)$ for all (x,y) "near" (a,b).

  That is, if there exists a disc C centered at (a,b) where
  $$f(x,y) \le f(a,b) \quad \text{for all}$$
  (x,y) in C and D.

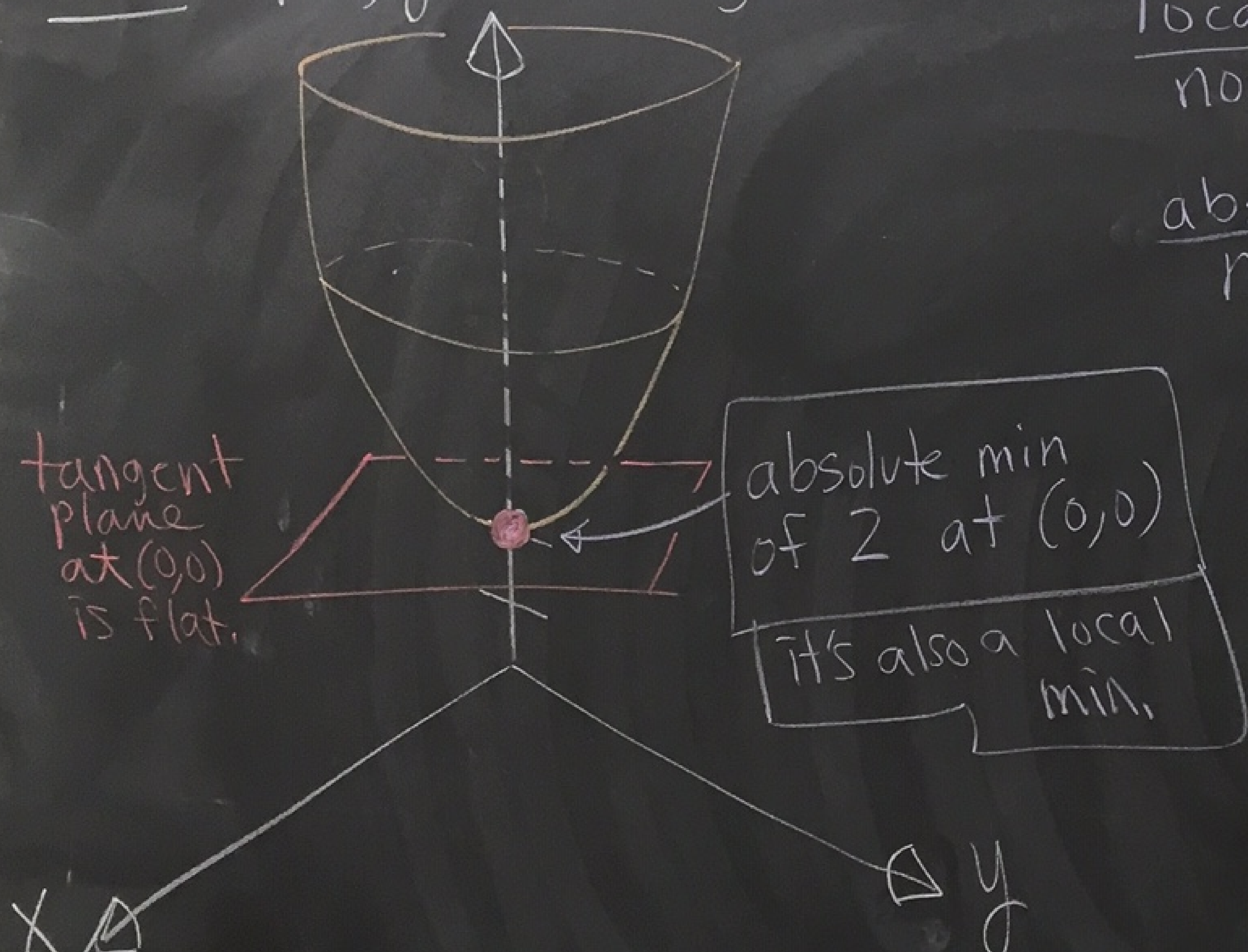


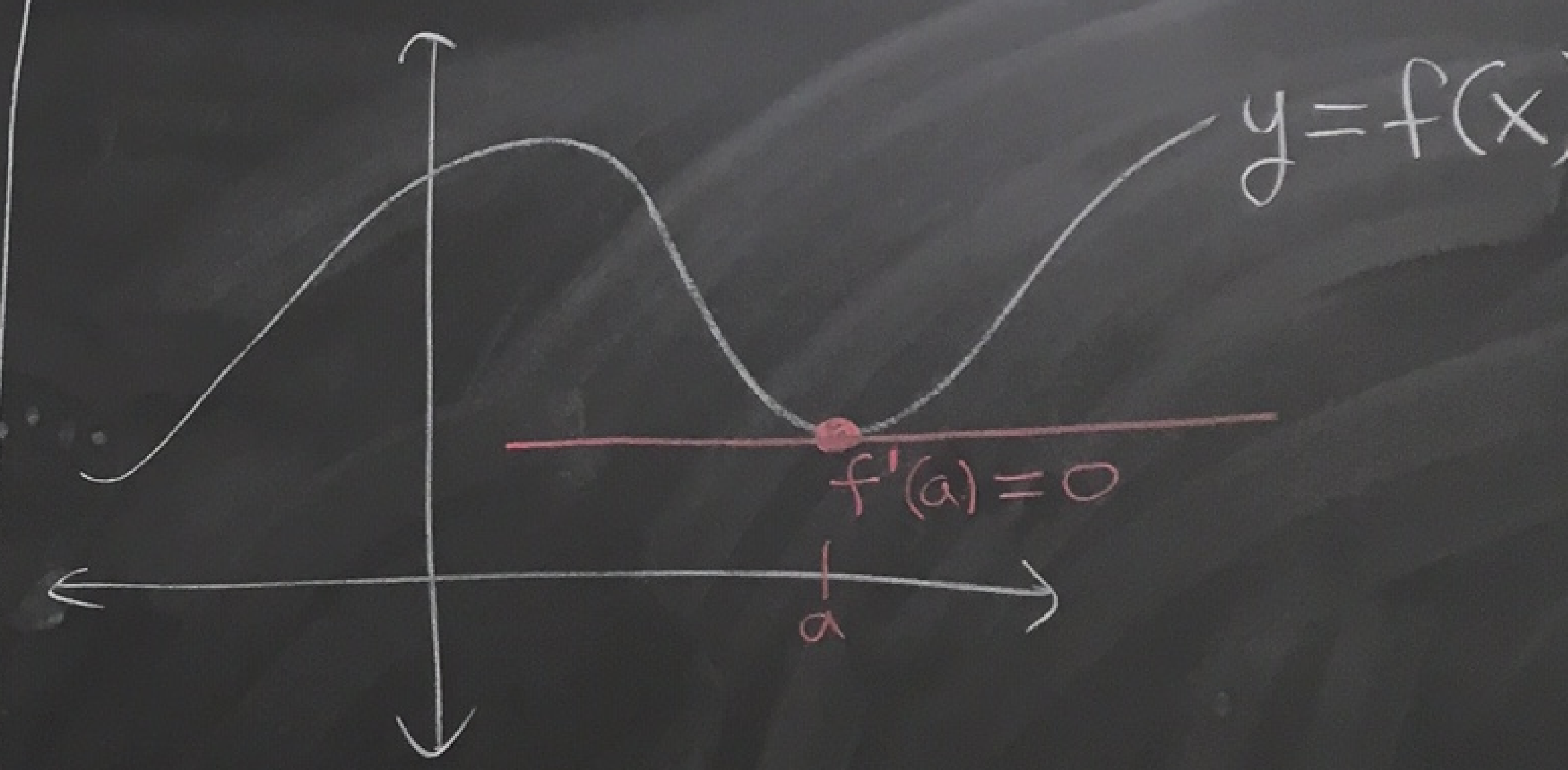
- (a,b) is a <u>local minimum</u> of f if
  $f(a,b) \le f(x,y)$ for all (x,y) "near" (a,b).
  That is, there exists a disc C centered at (a,b) where $f(a,b) \le f(x,y)$ for all (x,y) in C and D.
- (a,b) is an <u>absolute maximum</u> of f
  if $f(x,y) \le f(a,b)$ for all (x,y) in the domain D.
- (a,b) is an <u>absolute minimum</u> of f
  if $f(a,b) \le f(x,y)$ for all (x,y) in the domain D.

Ex: $f(x,y) = x^2 + y^2 + 2$

local max
none

absolute max
none

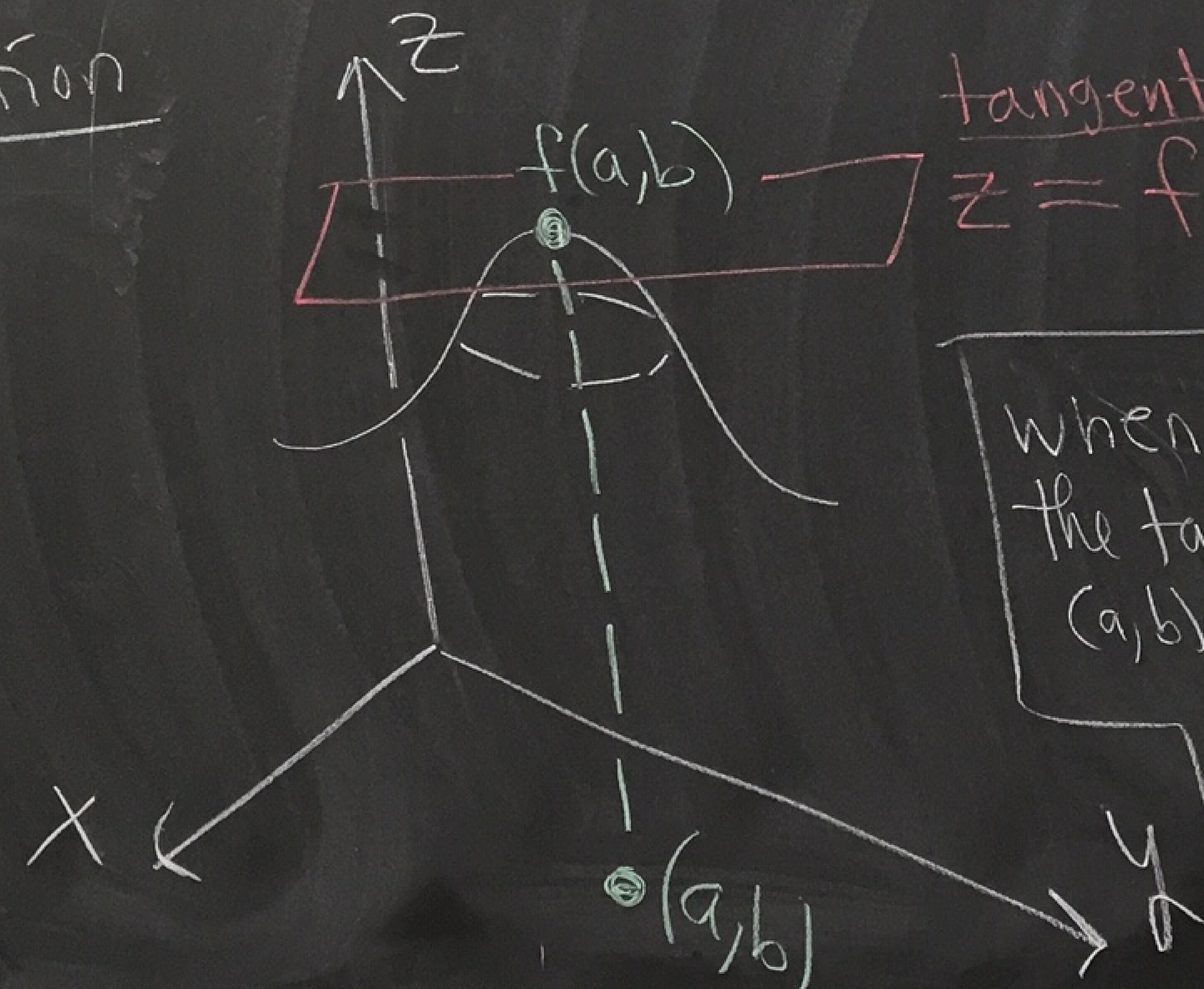




Theorem: If $f(x,y)$ has a local maximum or minimum at (a,b) and f_x and f_y exist at (a,b), then $f_x(a,b) = 0$ and $f_y(a,b) = 0$.

Def:

Intuition



tangent plane

$$z = f(a,b) + \underbrace{f_x(a,b)(x-a) + f_y(a,b)(y-b)}$$

has to be 0 to make the tangent plane flat; ie $f_x(a,b) = 0 = f_y(a,b)$

when $f_x(a,b) = 0 = f_y(a,b)$ the tangent plane at (a,b) has equation $z = f(a,b)$

$) + f_y(a,b)(y-b)$

be 0

e the tangent

flat; ie

$b) = 0 = f_y(a,b)$

Def: (a,b) is a critical point of $f(x,y)$ if either

① $f_x(a,b) = 0$ and $f_y(a,b) = 0$

or ② one of $f_x(a,b)$ or $f_y(a,b)$ does not exist.

By the theorem, local max's or local min's must be at critical points.

But a critical point might not be a local max or min.

Second Derivatives Test

Suppose the second order partial derivatives of $f(x,y)$ are continuous on a disc with center (a,b) and $f_x(a,b)=0$ and $f_y(a,b)=0$.

Let

$$D = f_{xx}(a,b)\, f_{yy}(a,b) - \left[f_{xy}(a,b)\right]^2$$

(a) If $D>0$ and $f_{xx}(a,b)>0$, then (a,b) is a local minimum.

(b) If $D>0$ and $f_{xx}(a,b)<0$, then (a,b) is a local maximum.

(c) If $D<0$, then (a,b) is neither a local minimum or local maximum.

here (a,b) is called a saddle point

- If $D=0$, the test gives no info.

- Can you get $D>0$ and $f_{xx}(a,b)=0$?? If so, then

$$\underbrace{D}_{>0} = \underbrace{-\left[f_{xy}(a,b)\right]^2}_{\leq 0}$$

CAN'T HAPPEN

How do you remember the formula for D?

$$D = \det\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$$

$$= f_{xx} f_{yy} - f_{xy} f_{yx}$$

$$= f_{xx} f_{yy} - f_{xy}^2$$

$f_{xy} = f_{yx}$

Ex: $f(x,y) = xy(x-2)(y+3)$

$$= (x^2-2x)(y^2+3y)$$

$$f_x = (y^2+3y)(2x-2) = 2y(y+3)(x-1)$$

$$f_y = (x^2-2x)(2y+3) = x(x-2)(2y+3)$$

$$f_{xx} = (y^2+3y)(2)$$

$$f_{yy} = (x^2-2x)(2)$$

$$f_{xy} = (2y+3)(2x-2)$$

Critical points

$0 = f_x = 2y(y+3)(x-1)$

AND $0 = f_y = x(x-2)(2y+3)$

$f_x = 0$ when $y=0$ or $y=-3$ or $x=1$

$f_y = 0$ when $x=0$ or $x=2$ or $y=-\frac{3}{2}$

Critical points Need $f_x=0$ AND $f_y=0$

$(a,b) = (0,0), (2,0), (0,-3), (2,-3), (1,-\frac{3}{2})$

Check $(0,0)$

$D = f_{xx} f_{yy} - (f_{xy})^2$

$f_{xx}(0,0) = 0$

$f_{yy}(0,0) = 0$

$f_{xy}(0,0) = -6$

$D = -(-6)^2 = -36 < 0$

$(0,0)$ is a saddle point