Thursday
9/19

Idea: Let G be a group and let $H \leq G$. We want to define a group operation on the set of left cosets G/H. We want the operation to be

$$(aH)(bH) = (ab)H$$

The only issue is that sometimes this operation isn't well-defined.

Ex: $D_6 = \{1, r, r^2, s, sr, sr^2\}$

$H = \langle s \rangle = \{1, s\}$

left cosets

$H = \{1, s\} = sH$

$rH = \{r, rs\} = \{r, sr^2\} = (sr^2)H$

$r^2H = \{r^2, r^2s\} = \{r^2, sr^{-2}\} = \{r^2, sr\} = (sr)H$

$D_6/H = \{H, rH, r^2H\}$

Let's try the operation out

$$(sH)(rH) = (sr)H$$

$$\begin{array}{cc} \shortparallel & \shortparallel \end{array}$$

$$(H)(sr^2H) = (1sr^2)H = sr^2H$$

NOT EQUAL

So the operation $(aH)(bH) = abH$ is not well defined on D_6/H.

It will turn out that the operation $(aH)(bH)=(ab)H$ is well-defined on G/H iff $H \trianglelefteq G$. Let's get a theorem about normal subgroups before we prove this.

(Subgroup normality theorem)

<u>Thm:</u> Let G be a group and $H \leq G$.
The following are equivalent:

① $H \trianglelefteq G$, that is $aH = Ha$ for all $a \in G$

② $aHa^{-1} \subseteq H$ for all $a \in G$ ← $\boxed{aHa^{-1} = \{aha^{-1} \mid h \in H\}}$

③ $aHa^{-1} = H$ for all $a \in G$ ← $\boxed{N_G(H) = \{a \in G \mid aHa^{-1} = H\} = G}$

Proof:

(① ⟹ ②) Assume $H \trianglelefteq G$.

Let $a \in G$.

We want to show $aHa^{-1} \subseteq H$.

Let $x \in aHa^{-1}$.

Then $x = aha^{-1}$ for some $h \in H$.

Since $Ha^{-1} = a^{-1}H$ and $ha^{-1} \in Ha^{-1}$

then $ha^{-1} \in a^{-1}H$.

So, $ha^{-1} = a^{-1}h'$ for some $h' \in H$.

Thus, $x = aha^{-1} = aa^{-1}h' = h' \in H$.

So, $aHa^{-1} \subseteq H$.

(② ⟹ ③) Assume $aHa^{-1} \subseteq H$ for all $a \in G$.

Let $b \in G$.

By assumption $bHb^{-1} \subseteq H$.

Let's show $H \subseteq bHb^{-1}$.

Pick $h \in H$.

Then $b^{-1}hb = b^{-1}h(b^{-1})^{-1} \in H$ by assumption.

So,

$$h = bb^{-1}hbb^{-1} = b(b^{-1}hb)b^{-1} \in bHb^{-1}.$$

Thus, $H \subseteq bHb^{-1}$.

Therefore, $bHb^{-1} = H$ for all $b \in G$.

$(3 \Rightarrow 1)$ Assume that $aHa^{-1} = H$ for all $a \in G$.

Let $b \in G$.

Let's show $bH = Hb$.

Let $x = bh \in bH$ for some $h \in H$.

By assumption $bhb^{-1} \in H$.

So, $bhb^{-1} = h'$ for some $h' \in H$.

Thus, $x = bh = h'b \in Hb$.

So, $bH \subseteq Hb$.

Similarly you can show $Hb \subseteq bH$.

So, $bH = Hb$. $\blacksquare$

(Well-defined coset operation theorem using normal subgroups)

Theorem: Let G be a group and $H \leq G$. The operation

$$(aH)(bH) = (ab)H$$

is well-defined on G/H iff $H \trianglelefteq G$.

proof:

($\Rightarrow$) Suppose the operation is well-defined.

Let $x \in G$.

We will show $\boxed{xHx^{-1} \subseteq H}$

This will imply that $H \trianglelefteq G$.

Let $\boxed{y = xhx^{-1} \in xHx^{-1}}$ where $h \in H$.

Since $h \in H$ we know $hH = 1H$. $\leftarrow$ $\boxed{\begin{array}{l} hH = 1H \\ \text{since } h \in 1H \end{array}}$

Since the operation is well-defined we have $(hH)(x^{-1}H) = (1H)(x^{-1}H)$.

So, $(hx^{-1})H = x^{-1}H$.

Note that $hx^{-1} = hx^{-1}1 \in hx^{-1}H$.

So, $hx^{-1} \in x^{-1}H$.

So, $hx^{-1} = x^{-1}h'$ for some $h' \in H$.

Thus, $\boxed{y = xhx^{-1} = xx^{-1}h' = h' \in H.}$

So, $xHx^{-1} \subseteq H$.

$aH = bH$
$cH = dH$
Show
$acH = bdH$

$ach = adh'$
$= ah''d$
$= bh'''d$
$= bdh''''$

($\Leftarrow$) Suppose H is normal in G.

Let's show the operation is well-defined.

Let $a, b, c, d \in G$.

Suppose $aH = bH$ and $cH = dH$.

We need to show that $(aH)(cH) = (bH)(dH)$, ie that $(ac)H = (bd)H$.

Let $x \in (ac)H$.

Then $x = ach$ for some $h \in H$.

Since $cH = dH$, we know $ch = dh'$ for some $h' \in H$.

Since $H \trianglelefteq G$ we know $dH = Hd$, so $dh' = h''d$ for some $h'' \in H$.

Since $aH=bH$, we know $ah''=bh'''$ for some $h'''\in H$.
Since $H \trianglelefteq G$ we have $Hd=dH$ and so $h'''d=dh''''$ for some $h''''\in H$.
Thus, $x=ach=adh'=ah''d=bh'''d=bdh''''\in(bd)H$.
Therefore, $(ac)H\subseteq(bd)H$.
Similarly you can show $(bd)H\subseteq(ac)H$.
So, $(bd)H=(ac)H$.

<u>Theorem:</u> Let G be a group and $H \trianglelefteq G$.
Then G/H is a group using the operation
$$(aH)(bH)=(ab)H.$$

<u>proof:</u> (closure) If $a,b\in G$, then $ab\in G$. So, $(ab)H$ is a coset.

(associativity) Let $a,b,c\in G$, then since G is associative $a(bc)=(ab)c$ and so
$$(aH)[(bH)(cH)]=(aH)((bc)H)=a(bc)H=(ab)cH=((ab)H)(cH)=[(aH)(bH)](cH).$$

(identity) Let 1 be the identity of G. Then
$(1H)(aH)=aH$ and $(aH)(1H)=aH$ for all $a\in G$.
So, $1H=H$ is the identity element of G/H.

(inverses) Let $a \in G$. Then a^{-1} exists. And

$$(aH)(a^{-1}H) = (aa^{-1})H = 1H = H$$

and $$(a^{-1}H)(aH) = (a^{-1}a)H = 1H = H$$

So, $(aH)^{-1} = a^{-1}H$.

$\blacksquare$