3.3 – Isomorphism Theorems

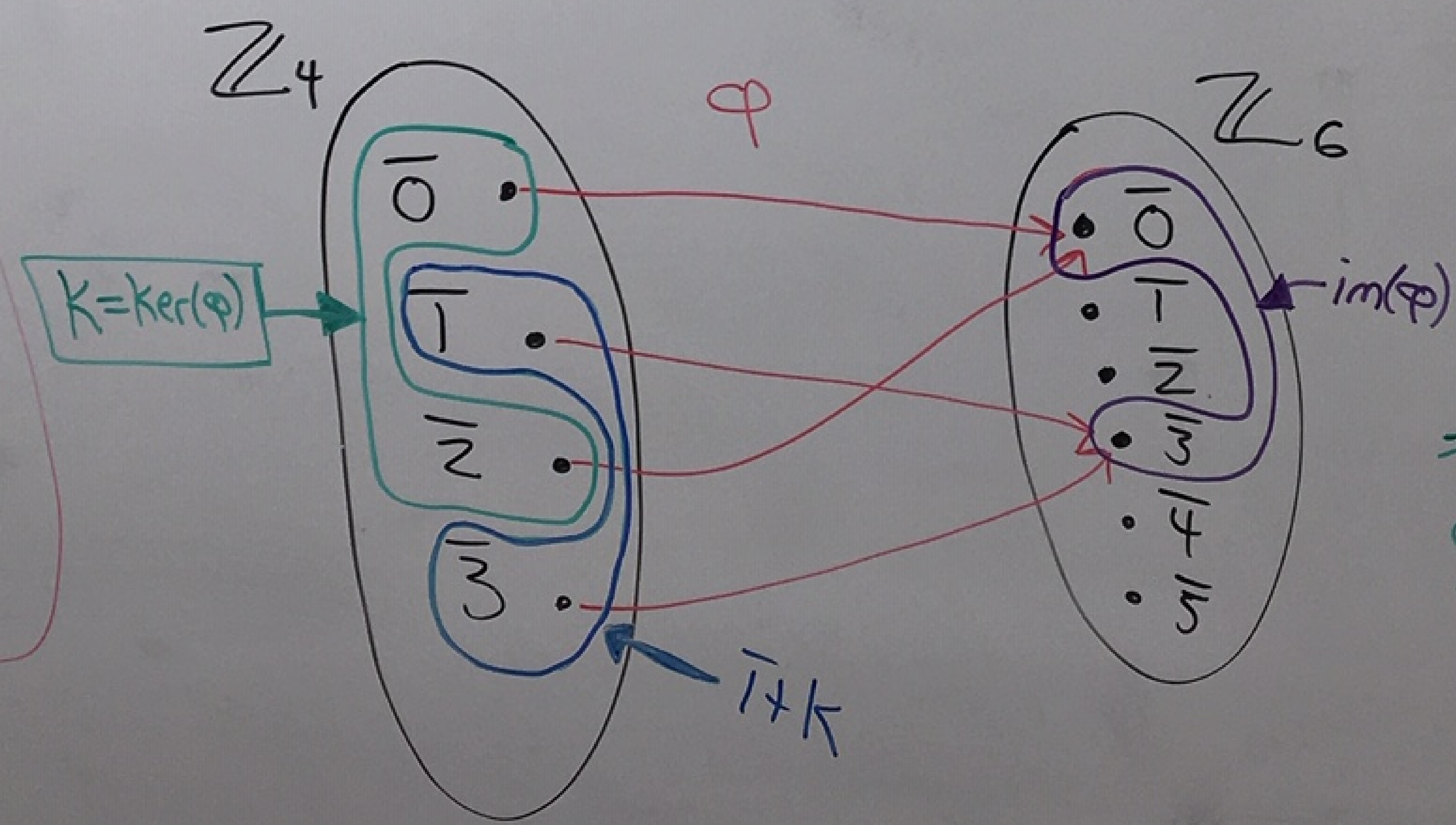
Constructing φ

If $\varphi(\bar{1})=\bar{3}$ then...

$$\varphi(\bar{2})=\varphi(\bar{1})+\varphi(\bar{1}) = \bar{3}+\bar{3}=\bar{0}$$

$$\varphi(\bar{3})=\varphi(\bar{1})+\varphi(\bar{1})+\varphi(\bar{1}) = \bar{3}+\bar{3}+\bar{3}=\bar{3}$$

Ex: Consider the following homomorphism.



$K=\ker(\varphi)=\{\bar{0},\bar{2}\}$

From last time: $K \trianglelefteq \mathbb{Z}_4$

So, $\mathbb{Z}_4/K$ is a group.

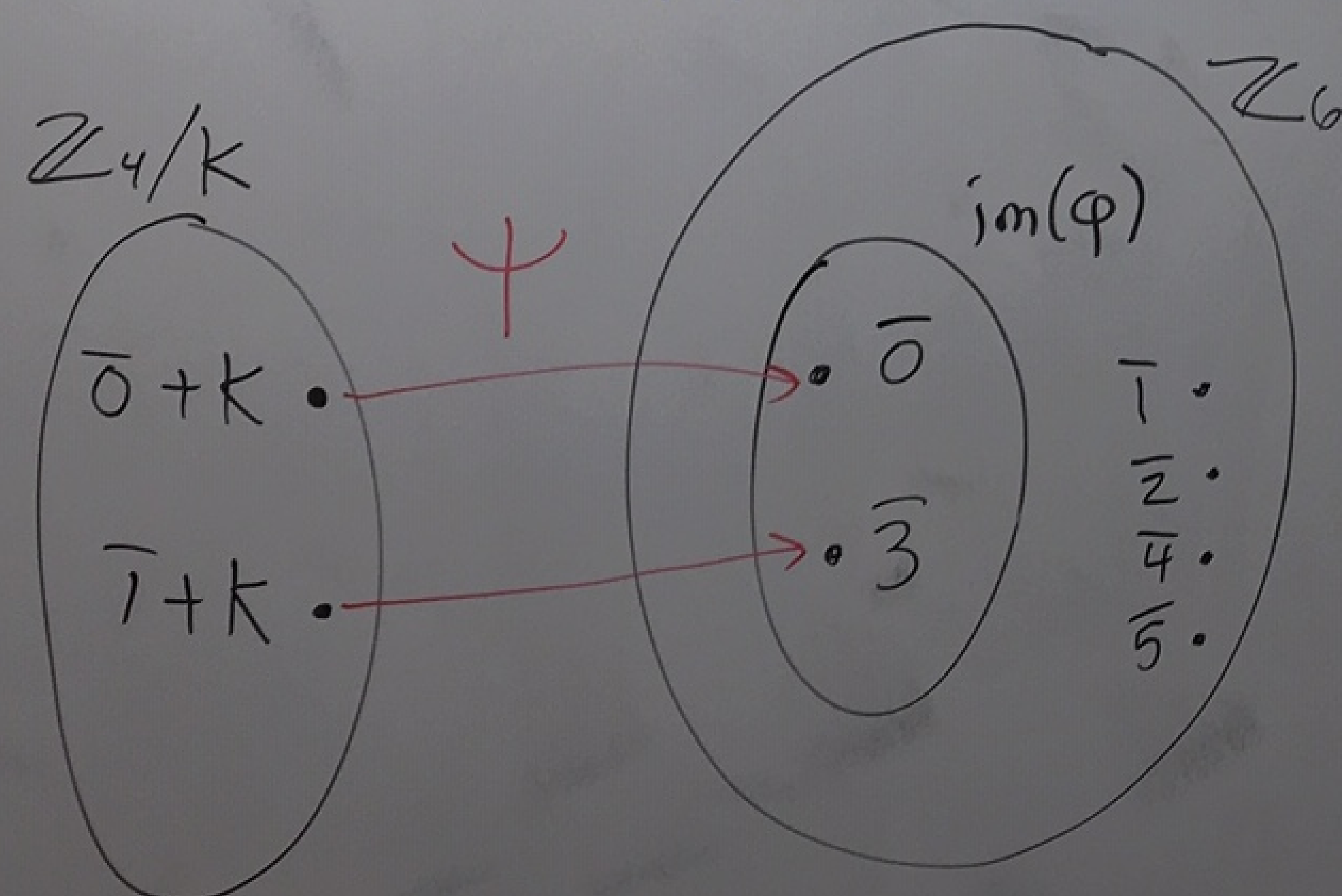
left cosets: $\bar{0}+K=\{\bar{0},\bar{2}\}$

$\bar{1}+K=\{\bar{1},\bar{3}\}$

$\trianglelefteq \mathbb{Z}_4$

oup.

Construct $\psi: \mathbb{Z}_4/K \to \operatorname{im}(\varphi)$ by $\psi(\bar{a}+K) = \varphi(\bar{a})$.



ψ will be an isomorphism between $\mathbb{Z}_4/K$ and $\operatorname{im}(\varphi) = \langle \bar{3} \rangle = \{\bar{0}, \bar{3}\} \cong \mathbb{Z}_2$

Theorem: (The First Isomorphism Theorem) Let G and G' be groups. Let $\varphi: G \to G'$ be a homomorphism. Then

1. $\ker(\varphi) \trianglelefteq G$

and 2. $G/\operatorname{Ker}(\varphi) \cong \operatorname{im}(\varphi)$

Proof: From earlier in the class we proved that $\ker(\varphi) \trianglelefteq G$. In HW you showed $im(\varphi) = \varphi(G) \leq G'$.

Let's prove part ②. Let $K = \ker(\varphi)$.

Define $\psi : G/K \to im(\varphi)$ by $\psi(gK) = \varphi(g)$.

$\boxed{\psi \text{ is well-defined}}$

Suppose $g_1K = g_2K$ where $g_1, g_2 \in G$. We need to show that $\psi(g_1K) = \psi(g_2K)$.

We know that $\psi(g_1K) = \varphi(g_1)$ and $\psi(g_2K) = \varphi(g_2)$.

Since $g_1K = g_2K$ we have that $g_1 \in g_2K$.

So, $g_1 = g_2k$ where $k \in \ker(\varphi)$.

So,

$\psi(g_1K) = \varphi(g_1) = \varphi(g_2k)$

$= \varphi(g_2)\varphi(k)$

$= \varphi(g_2)\cdot 1_{G'}$ ← $k \in \ker(\varphi)$

$= \varphi(g_2) = \psi(g_2K)$.

$aH = bH$
iff
$b^{-1}a \in H$
iff
$a \in bH$

$\boxed{\psi \text{ is a homomorphism}}$

Let $xK, yK \in G/K$ where $x, y \in G$.

Then, $\psi((xK)(yK)) = \psi((xy)K)$

$= \varphi(xy)$

$= \varphi(x)\varphi(y) = \psi(xK)\psi(yK)$ ← φ is a hom.

$\boxed{\psi \text{ is } 1\text{-}1}$

Suppose $\psi(xK) = \psi(yK)$ for some $x, y \in G$.

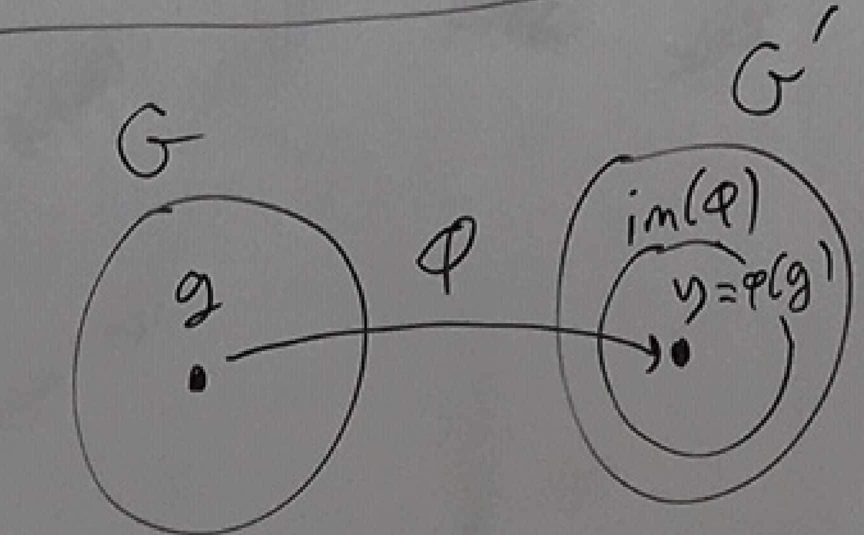
So, $\varphi(x) = \varphi(y)$.

Thus, $\varphi(y)^{-1}\varphi(x) = \varphi(y)^{-1}\varphi(y)$.

So, $\varphi(y^{-1}x) = 1_{G'}$.

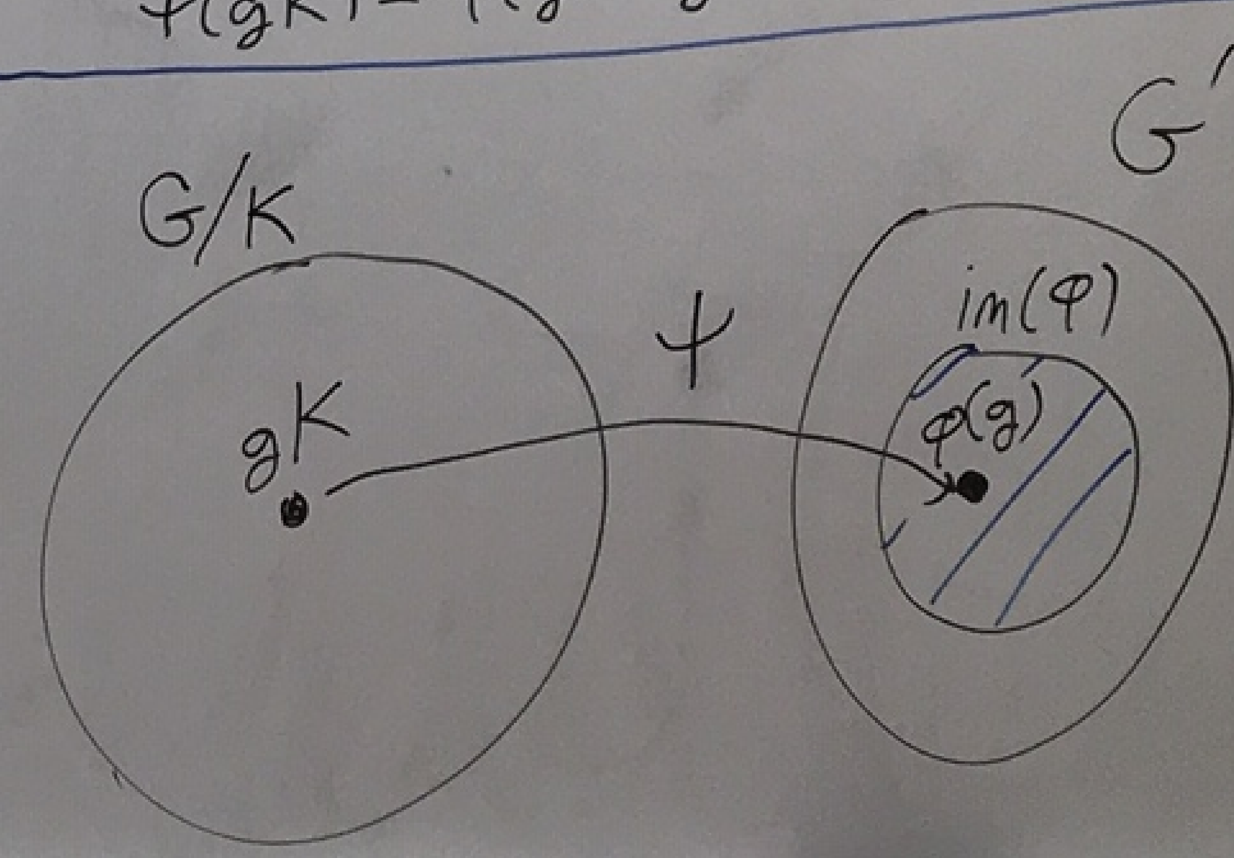
Thus, $y^{-1}x \in K = \ker(\varphi)$.

So, $yK = xK$.



$\boxed{\psi \text{ is onto } \operatorname{im}(\varphi)}$

Let $y \in \operatorname{im}(\varphi)$. Then $y = \varphi(g)$ for some $g \in G$. Then $gK \in G/K$ and $\psi(gK) = \varphi(g) = y$.



Therefore, ψ is an isomorphism between G/K and $\mathrm{im}(\varphi)$.

So, $G/K \cong \mathrm{im}(\varphi)$.

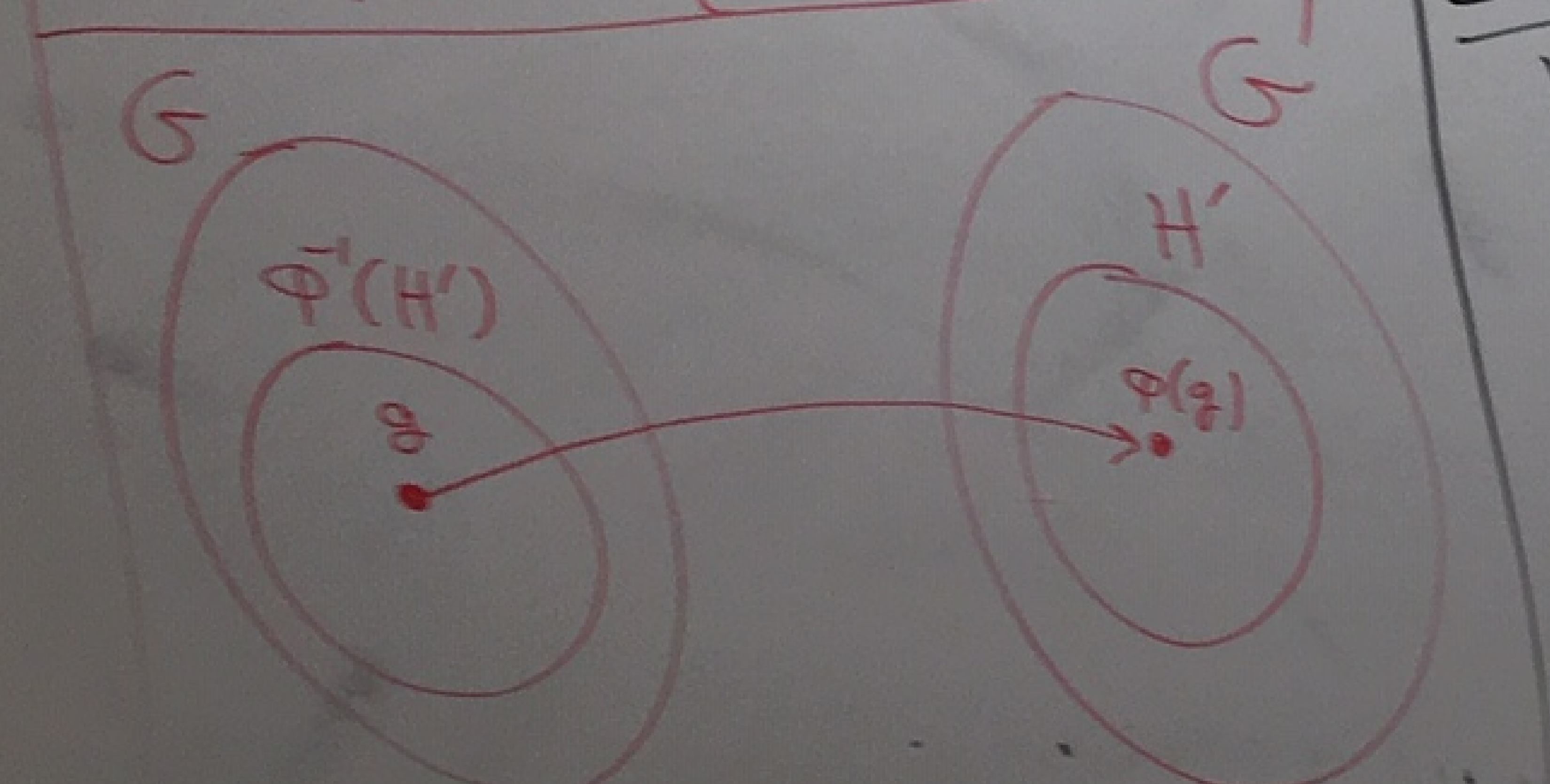
$\blacksquare$

Lemma: Let $\varphi : G \to G'$ be a homomorphism between two groups G and G'.

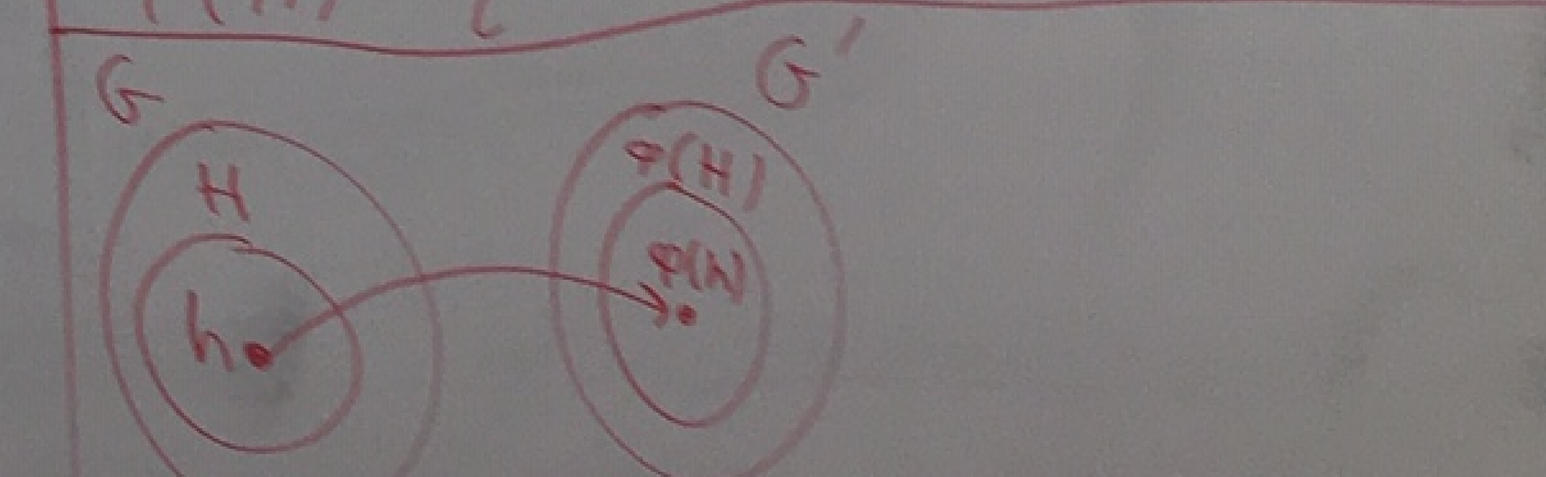
① If $H \leq G$, then $\varphi(H) \leq G'$

② If $H' \leq G'$, then $\varphi^{-1}(H') \leq G$.

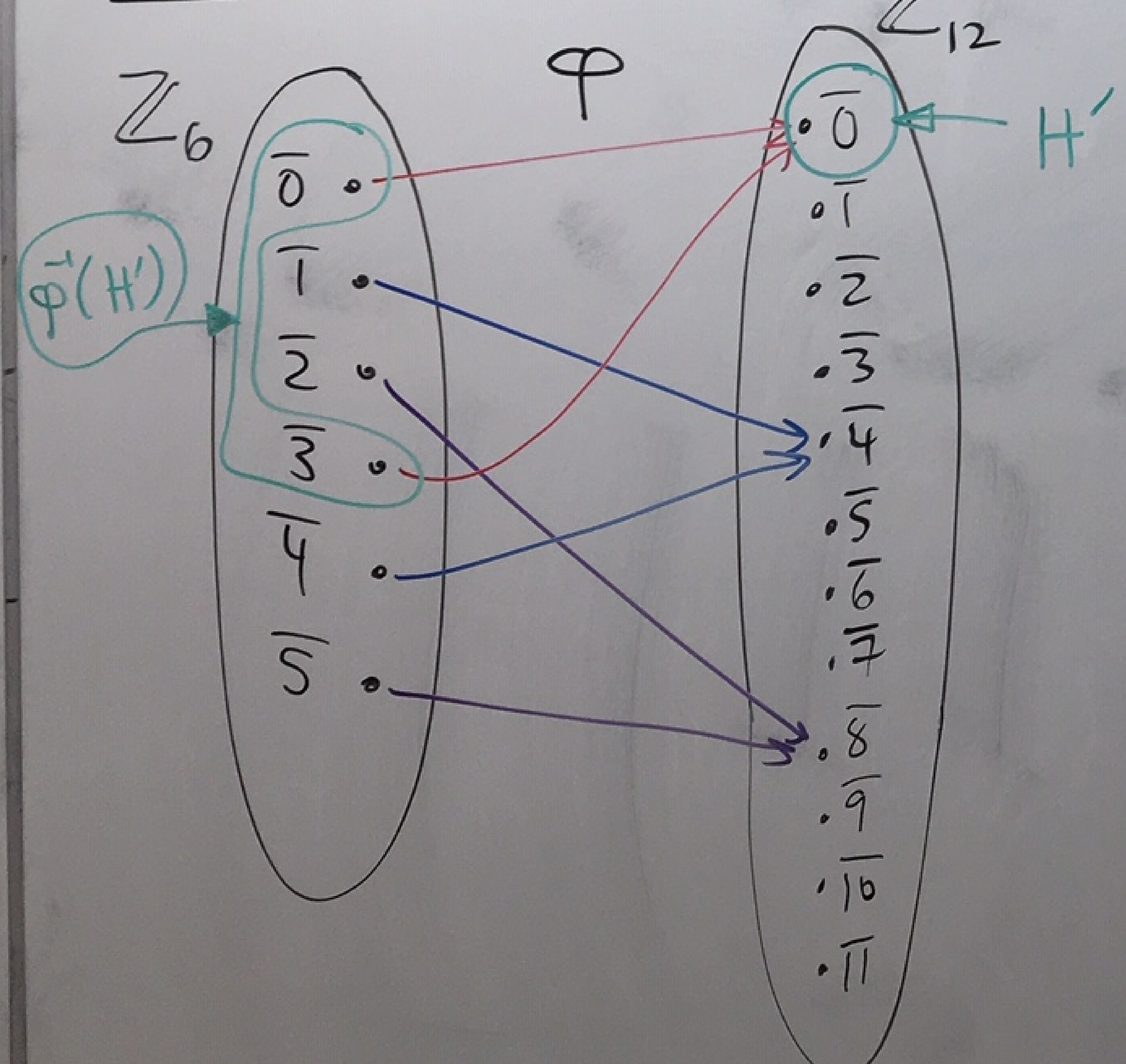
$\varphi^{-1}(H') = \{g \in G \mid \varphi(g) \in H'\}$



$\varphi(H) = \{\varphi(h) \mid h \in H\}$



Ex:



$$\varphi^{-1}(\{\bar0\}) = \{\bar0, \bar3\} = \langle \bar3 \rangle \leq \mathbb{Z}_6$$

$$\begin{aligned}
\varphi(\langle \bar2 \rangle) &= \varphi(\{\bar0, \bar2, \bar4\}) \\
&= \{\varphi(\bar0), \varphi(\bar2), \varphi(\bar4)\} \\
&= \{\bar0, \bar8, \bar4\} = \langle \bar4 \rangle \leq \mathbb{Z}_{12}
\end{aligned}$$

Scratchwork

Goal:

$\varphi(xy^{-1}) \in H'$

or $\varphi(x)\varphi(y)^{-1} \in H'$

Proof of ② Let $H' \leq G'$.

We want to show that $\varphi^{-1}(H') = \{x \in G \mid \varphi(x) \in H'\}$ is a subgroup of G.

- Since $H' \leq G'$ we know $1_{G'} \in H'$.
  And since φ is a homomorphism $\varphi(1_G) = 1_{G'}$.
  So, $1_G \in \varphi^{-1}(H')$
- Let $x, y \in \varphi^{-1}(H')$.
  So, $\varphi(x) \in H'$ and $\varphi(y) \in H'$.
  Since $H' \leq G'$ we know $\varphi(y)^{-1} \in H'$.
  Also since $\varphi(x) \in H'$ and $\varphi(y)^{-1} \in H'$ we have $\varphi(x)\varphi(y)^{-1} \in H'$.
  Thus, $\varphi(xy^{-1}) = \varphi(x)\varphi(y)^{-1} \in H'$. So, $xy^{-1} \in \varphi^{-1}(H')$.

roup of G.

Theorem: Let G be a group and $K \trianglelefteq G$.
Let $\varphi : G \to G/K$ be the canonical/natural homomorphism. That is, $\varphi(g) = gK$.
Then φ gives a one-to-one correspondence between the following two sets:

$$\{H \leq G \mid K \subseteq H\} \xleftrightarrow{\ \varphi\ } \{\overline{H} \mid \overline{H} \leq G/K\}$$

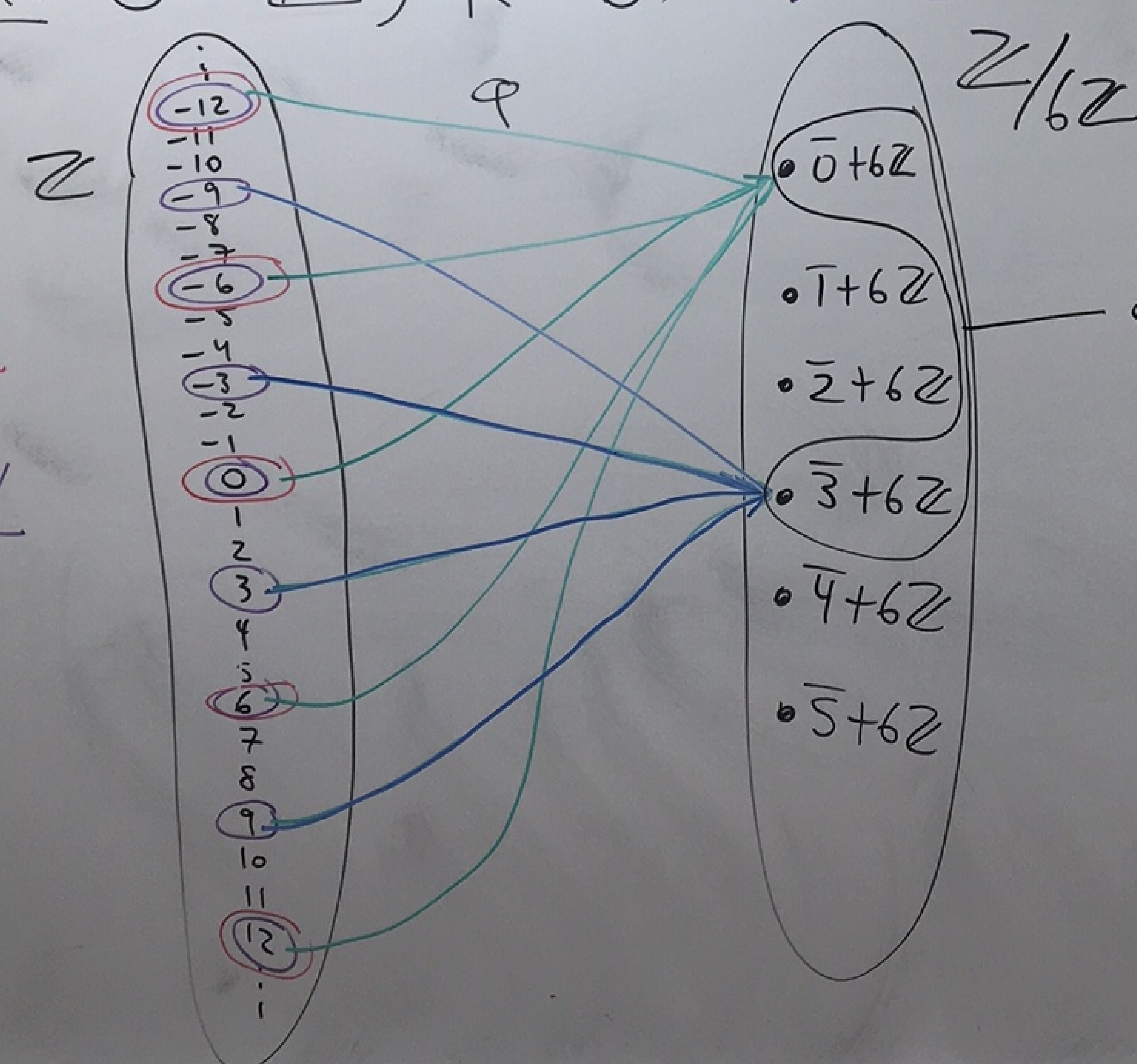
$$H \longmapsto \varphi(H)$$

$$\varphi^{-1}(H) \longleftarrow\!\shortmid \widehat{H}$$

Ex: $G = \mathbb{Z}$, $K = 6\mathbb{Z}$, $H = 3\mathbb{Z}$

$\bigcirc = 6\mathbb{Z}$

$\bigcirc = 3\mathbb{Z}$



$\varphi(H) = \{\bar{0}+6\mathbb{Z}, \bar{3}+6\mathbb{Z}\} = \langle \bar{3}+6\mathbb{Z} \rangle \leq \mathbb{Z}/6\mathbb{Z}$

If $\bar{H} = \{\bar{0}+6\mathbb{Z}, \bar{3}+6\mathbb{Z}\} \leq \mathbb{Z}/6\mathbb{Z}$

then $\varphi^{-1}(\bar{H}) = 3\mathbb{Z}$

and $K \subseteq 3\mathbb{Z}$.