Fall 2023 Algebra Comprehensive Exam — Solutions

Linear Algebra

(L1) Let V be a vector space over a field $\mathbb{F}$. Let W be a subspace of V. Fix an element $v_0 \in V$. Define the set $W_0 = \{v_0 + w : w \in W\}$. Prove that W_0 is a subspace of V if and only if $v_0 \in W$.

`Answer:` *Assume W_0 is a subspace. Then, $\overrightarrow{0} \in W_0$. That is, there exists $w \in W$ such that $\overrightarrow{0} = v_0 + w$. So, $v_0 = -w \in W$. Conversely, if $v_0 \in W$ then $W_0 \neq \emptyset$: $v_0 = v_0 + \overrightarrow{0} \in W_0$. And if $w, w' \in W$, $(v_0 + w) + (v_0 + w') = v_0 + (w + v_0 + w') \in W_0$. Therefore, W_0 is a subspace of V.*

(L2) Let V be the real vector space of real functions spanned by $\{1, x, e^x\}$, and let $h : V \to V$ be defined by $h(f) = f - f'$ for all $f \in V$, that is, $h(f)$ is f minus its derivative. No need to prove that h is a linear function.

(1) What is the dimension of V?

(2) Find a basis for the space $\ker h = \{f \in V : h(f) = 0\}$.

(3) Find a basis for the space $\operatorname{im} h = \{h(f) : f \in V\}$.

`Answer:`

(1) *We show that $\{1, x, e^x\}$ is linearly independent and hence $\dim V = 3$. Suppose that $c_1 + c_2 x + c_3 e^x = 0$ for some $c_1, c_2, c_3 \in \mathbb{R}$. Plugging in $x = -1$, $x = 0$ and $x = 1$ into this equation gives the system*

$$\begin{aligned} c_1 - c_2 + c_3 e^{-1} &= 0 \\ c_1 + c_3 &= 0 \\ c_1 + c_2 + c_3 e &= 0 \end{aligned}$$

which can be solved to give $c_1 = c_2 = c_3 = 0$.

Or, since differentiation is a linear function, setting $x = 0$ into $c_1 + c_2 x + c_3 e^x = 0$ and the first and second derivatives of this equation gives $c_1 + c_3 = 0$, $c_2 + c_3 = 0$ and $c_3 = 0$, which is even easier to solve to get $c_1 = c_2 = c_3 = 0$.

(2) *If $f = c_1 + c_2 x + c_3 e^x$ for some $c_1, c_2, c_3 \in \mathbb{R}$, then $h(f) = (c_1 + c_2 x + c_3 e^x) - (c_2 + c_3 e^x) = (c_1 - c_2) + c_2 x$. So f is in $\ker h$ if and only if $c_1 - c_2 = c_2 = 0$ if and only if $c_1 = c_2 = 0$, if and only if $f = c_3 e^x$ for some $c_3 \in \mathbb{R}$. So a basis for $\ker h$ is $\{e^x\}$.*

(3) *Since, from above, $h(f) = (c_1 - c_2) + c_2 x$, $\{1, x\}$ is a basis for $\operatorname{im} h$.*

(L3) Let T be an arbitrary linear operator on a vector space V, and let λ and μ be two distinct eigenvalues of T.

(a) Prove or disprove: If v is an eigenvector of T with eigenvalue λ, and w is an eigenvector of T with eigenvalue μ, then $v + w$ is an eigenvector of T.

(b) Prove or disprove: If v and w are eigenvectors of T with eigenvalue λ, then $v + w$ is an eigenvector of T.

`Answer:` *The statement in (a) is false. For instance, if $V = \mathbb{R}^2$ and $T(x, y) = (x, 2y)$, then $(1, 0)$ has eigenvalue 1, and $(0, 1)$ has eigenvalue 2. But $(1, 0) + (0, 1) = (1, 1)$, and $(1, 1)$ is not an eigenvector since $T(1, 1) = (1, 2)$ is not a scalar multiple of $(1, 1)$.*

But (b) is true: $T(v + w) = Tv + Tw = \lambda v + \lambda w = \lambda(v + w)$, so $v + w$ is an eigenvector with eigenvalue λ.

Groups

(G1) Let $G = \mathbb{Z}_4 \times \mathbb{Z}_6$ and $H = \langle (2,2) \rangle$. What familiar group is G/H isomorphic to?

`Answer:` *We have $|G| = 24$ and $|H| = 6$, and so $|G/H| = 4$. Hence G/H is isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$. The elements of G/H are*

$$\begin{aligned} H &= \{(0,0),(2,2),(0,4),(2,0),(0,2),(2,4)\} \\ (1,0)+H &= \{(1,0),(3,2),(1,4),(3,0),(1,2),(3,4)\} \\ (1,1)+H &= \{(1,1),(3,3),(1,5),(3,1),(1,3),(3,5)\} \\ (0,1)+H &= \{(0,1),(2,3),(0,5),(2,1),(0,3),(2,5)\} \end{aligned}$$

Since $(1,0)+(1,0) = (2,0) \in H$, $(1,1)+(1,1) = (2,2) \in H$ and $(0,1)+(0,1) = (0,2) \in H$, all nonidentity elements of G/H have order 2 and so $G/H \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

(G2) Let G be a group. For $x, y \in G$, recall that x is *conjugate* to y in G if there exists $g \in G$ such that $y = gxg^{-1}$. We will write $x \sim y$ to denote that x is conjugate to y in G. Prove that the relation "$\sim$" is an equivalence relation on G.

`Answer:` *Let $x, y, z \in G$. The relation is reflexive because $1\,x\,1^{-1} = x$, so $x \sim x$. If $x \sim y$, then $y = gxg^{-1}$ for some g, so $x = g^{-1}y(g^{-1})^{-1}$, and so $y \sim x$; thus the relation is symmetric. And if $x \sim y$ and $y \sim z$, then $y = gxg^{-1}$ and $z = hyh^{-1}$ for some g, h, so $z = hgxg^{-1}h^{-1} = (hg)x(hg)^{-1}$, and so $x \sim z$; thus the relation is transitive.*

(G3) Let G be an Abelian group. Let H be the set of elements of G with finite order; that is, $H = \{g \in G : |g| < \infty\}$. Prove that H is a subgroup of G.

`Answer:` *First of all, the identity of G has order 1, so it is in H. Now let $g, h \in H$. This means that g and h have finite order; say $|g| = m$ and $|h| = n$. Then gh has finite order, because $(gh)^{mn} = g^{mn}h^{mn}$ (since G is Abelian), and $g^{mn} = (g^m)^n = 1$ and $h^{mn} = (h^n)^m = 1$. Thus $gh \in H$. Finally, g^{-1} has the same order as g, so $g^{-1} \in H$.*

Synthesis

(S1) Find a subgroup of $GL_2(\mathbb{C})$ that is isomorphic to $\mathbb{Z}_4$.

`Answer:` *There are lots of answers. For example,*

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\}$$

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} i & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -i & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

(S2) Let H be the set of all matrices in $GL_2(\mathbb{R})$ with integer entries. Is H a subgroup of $GL_2(\mathbb{R})$? Prove your answer.

`Answer:` *No, because it is not closed under inverses: for instance, if $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, then $A \in H$ but $A^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \notin H$.*

(S3) Let n be a nonnegative integer. Recall that $\mathbb{R}^*$ is the group of non-zero real numbers under multiplication. Define $\phi : GL_n(\mathbb{R}) \to \mathbb{R}^*$ by $\phi(A) = \det(A)$, the determinant of A.

(a) Use a property of determinants to explain why ϕ is a homomorphism.

(b) Determine $\ker(\phi)$ and $\operatorname{im}(\phi)$.

(c) Based on your answers to (a) and (b), use the First Isomorphism Theorem to make a statement involving a quotient group of $GL_n(\mathbb{R})$.

`Answer:` *(a) Note that* $\phi(A \cdot B) = \det(A \cdot B) = \det(A) \cdot \det(B) = \phi(A) \cdot \phi(B)$. *Therefore,* ϕ *is a homomorphism.*

(b) $\ker(\phi) = \{A \in GL_n(\mathbb{R}) : \ det(A) = 1\} = SL_n(\mathbb{R})$.

$\operatorname{im}(\phi) = \mathbb{R}^*$, *because, for each* $r \in \mathbb{R}^*$, *the diagonal matrix* A *with entries* $r, 1, 1, \ldots, 1$ *has determinant* r, *so* $\phi(A) = r$.

(c) By the First Isomorphism Theorem, $GL_n(\mathbb{R})/\ker(\phi)$ *is isomorphic to* $\mathbb{R}^*$.