California State University – Los Angeles

Department of Mathematics

Master's Degree Comprehensive Examination

Analysis   Spring 2024

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Do at least two (2) problems from Section 1 below, and at least three (3) problems from Section 2 below. All problems count equally. If you attempt more than two problems from Section 1, the best two will be used. If you attempt more than three problems from Section 2, the best three will be used. Be sure to show your work for all answers.

(1) Write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.
(2) Write on one side of the paper only.
(3) Begin each problem on a new page.
(4) Assemble the problems you hand in in numerical order.

Exams are graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

SECTION 1 – Do two (2) problems from this section. If you attempt all three, then the best two will be used for your grade.

Spring 2024 #1. Let $x_1 > 1$ and

$$x_{n+1} = 2 - \frac{1}{x_n}$$

for $n \in \mathbb{N}$. Show that the sequence x_n is bounded and monotone. Find its limit.

Spring 2024 #2. Let S be a discrete subset of $\mathbb{R}$. Prove that S is compact if and only if S contains only finitely many elements. (To say that S is "discrete" means that for all $x \in S$, there exists $r > 0$ such that $S \cap (x - r, x + r) = \{x\}$.)

Spring 2024 #3. Let x_n be a sequence of real numbers.

(a) State the definition of a Cauchy sequence.

(b) Prove that if the sequence x_n converges, then it is a Cauchy sequence.

SECTION 2 – Do three (3) problems from this section. If you attempt more than three, then the best three will be used for your grade.

Spring 2024 #4. Let $\mathcal{H}$ be a complex Hilbert space, and let $y \in \mathcal{H}$. Let T be the bounded linear transformation $T : \mathcal{H} \to \mathbb{C}$ defined by

$$T(x) = \langle y, x \rangle.$$

(a) Show that T is a bounded linear operator.

(b) Find the operator norm of T.

Spring 2024 #5. For each $n \in \mathbb{N}$, let $f_n : [0,1] \to \mathbb{R}$ be defined by $f_n(x) = x^n$. Find the norm of f_n in the following spaces:

(a) $C([0,1])$, with the norm $\|f\|_{C([0,1])} = \sup_{x \in [0,1]} |f(x)|$.
(b) $L^1([0,1])$, with the standard L^1 norm.

Spring 2024 #6. Let X be a Banach space. (Recall that this means that X is a normed vector space that is complete with respect to the metric induced by that norm.) Let S be a closed linear subspace of X. Prove that S is a Banach space. (For the norm on S, take the restriction to S of the norm on X.)

Spring 2024 #7. Let X be an inner product space over $\mathbf{R}$. Show that two vectors $x, y \in X$ are orthogonal if and only if

$$\|x + \alpha y\| = \|x - \alpha y\|$$

for every $\alpha \in \mathbf{R}$.