California State University – Los Angeles

Department of Mathematics

Master's Degree Comprehensive Examination

Analysis Spring 2024

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Do at least two (2) problems from Section 1 below, and at least three (3) problems from Section 2 below. All problems count equally. If you attempt more than two problems from Section 1, the best two will be used. If you attempt more than three problems from Section 2, the best three will be used. Be sure to show your work for all answers.

(1) Write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.
(2) Write on one side of the paper only.
(3) Begin each problem on a new page.
(4) Assemble the problems you hand in in numerical order.

Exams are graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

SECTION 1 – Do two (2) problems from this section. If you attempt all three, then the best two will be used for your grade.

Spring 2024 #1. Let $x_1 > 1$ and

$$x_{n+1} = 2 - \frac{1}{x_n}$$

for $n \in \mathbb{N}$. Show that the sequence x_n is bounded and monotone. Find its limit.

Proof. First, let us show the sequence x_n is bounded below by 1 for $n \geq 1$. We prove this by induction. By assumption, $x_1 > 1$. Next, assume that $x_k > 1$. Then

$$\frac{1}{x_k} < 1.$$

Then

$$x_{k+1} = 2 - \frac{1}{x_k} \geq 1.$$

By the principle of induction, tt follows x_n is bounded below by 1 for all n.

Next, we show that x_n is monotone decreasing. It suffices to show that

$$x_n - x_{n+1} \geq 0.$$

To see why this is the case, observe that

$$x_n - x_{n+1} = x_n + \left(2 - \frac{1}{x_n}\right) = x_n + \frac{1}{x_n} - 2.$$

To see why this must be non-negative: consider the function

$$f(x) = x + \frac{1}{x} - 2.$$

This function satisfies

$$f(1) = 0$$

and

$$f''(x) \geq 0.$$

Thus this function is increasing on the interval $[0, +\infty)$. It follows that $0 = f(1) \leq f(x)$ for all $x \geq 1$. It follows that

$$x_n + \frac{1}{x_n} - 2 \geq 0.$$

Thus x_n is monotone decreasing.

Since x_n is monotone decreasing and bounded below, it follows from the Monotone Convergence Theorem that it converges. Let x be its limit. Taking the limit on both sides of the definition of x_{n+1}, we have

$$x = 2 - \frac{1}{x},$$

which can be rewritten as

$$x^2 - 2x - 1 = 0.$$

This has solution $x = 1$. It follows that

$$\lim x_n = 1.$$

□

Spring 2024 #2. Let S be a discrete subset of $\mathbb{R}$. Prove that S is compact if and only if S contains only finitely many elements. (To say that S is "discrete" means that for all $x \in S$, there exists $r > 0$ such that $S \cap (x - r, x + r) = \{x\}$.)

Many proofs are possible. Here's one:

One direction is immediate, because every finite set is compact. (To say that a set is "finite" means that it contains only finitely many elements.)

Now assume that S is compact. We will show that S is finite.

We will prove this by contradiction. Temporarily assume that S is infinite. Let $\{x_n\}$ be a sequence of distinct points in S. Because S is compact, by the Bolzano-Weierstrass theorem, $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$. Then there exists $x \in S$ such that

$$\lim_{k\to\infty} x_{n_k} = x.$$

Because S is discrete, there exists $r > 0$ such that $S \cap (x-r, x+r) = \{x\}$. However, by def. of limit, there exists K such that $x_{n_k} \in (x-r, x+r)$ whenever $k \geq K$. In particular, that means that $x_{n_K} \in S \cap (x-r, x+r)$ and $x_{n_{K+1}} \in S \cap (x-r, x+r)$. Because of how we chose $\{x_n\}$, we know that $x_{n_K} \neq x_{n_{K+1}}$. This means that $S \cap (x-r, x+r)$ contains at least two distinct elements. Therefore $S \cap (x-r, x+r) \neq \{x\}$. Contradiction. Therefore S is finite.

Spring 2024 #3. Let x_n be a sequence of real numbers.

(a) State the definition of a Cauchy sequence.
(b) Prove that if the sequence x_n converges, then it is a Cauchy sequence.

Solution. (a) The sequence x_n is Cauchy if, for every $\epsilon > 0$, there exists a natural number N such that

$$|x_n - x_m| < \epsilon$$

whenever $n, m \geq N$.

(b) Assume x_n is a convergent sequence, and let $\epsilon > 0$. Then there exists a real number L and a natural number N such that

$$|x_n - L| < \frac{\epsilon}{2}$$

whenever $n \geq N$. Suppose now that $n, m \geq N$. We then have

$$\begin{aligned} |x_n - x_m| &= |x_n - L + L - x_m| \\ &\leq |x_n - L| + |L - x_m| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

It follows that x_n is a Cauchy sequence. □

SECTION 2 – Do three (3) problems from this section. If you attempt more than three, then the best three will be used for your grade.

Spring 2024 #4. Let $\mathcal{H}$ be a complex Hilbert space, and let $y \in \mathcal{H}$. Let T be the bounded linear transformation $T : \mathcal{H} \to \mathbb{C}$ defined by

$$T(x) = \langle y, x \rangle.$$

(a) Show that T is a bounded linear operator.

Proof. It suffices to show that there exists a constant $C > 0$ such that

$$|T(x)| \leq C\|x\|_{\mathcal{H}}$$

for all $x \in \mathcal{H}$. By the Cauchy-Schwarz inequality, we have that

$$\begin{aligned} |T(x)| &= |\langle y, x \rangle| \\ &\leq \|y\|_{\mathcal{H}}\|x\|_{\mathcal{H}}. \end{aligned}$$

Thus we may choose $C = \|y\|_{\mathcal{H}}$. It follows that T is bounded. □

(b) Find the operator norm of T.

Proof. Recall that the norm of T is given by

$$\|T\| = \inf\{C : |T(x)| \leq C\|x\|_{\mathcal{H}}\}.$$

From part (a), we know that

$$|T(x)| \leq \|y\|_{\mathcal{H}}\|x\|_{\mathcal{H}}.$$

Thus, we conclude that

$$\|T\| \leq \|y\|_{\mathcal{H}}.$$

To show the inequality in the other direction, we use the alternative characterization

$$\|T\| = \sup\{|T(x)| : \|x\|_{\mathcal{H}} \leq 1\}.$$

Applying the Cauchy-Schwartz inequality again for $\|x\|_{\mathcal{H}} \leq 1$, we obtain

$$\begin{aligned} |T(x)| &\leq \|y\|_{\mathcal{H}}\|x\|_{\mathcal{H}} \\ &\leq \|y\|_{\mathcal{H}}. \end{aligned}$$

Since $\|T\|$ is the supremum of all such numbers, we must have

$$\|y\|_{\mathcal{H}} \leq \|T\|.$$

Combining this with the previous inequality, we see that

$$\|T\| = \|y\|_{\mathcal{H}}.$$

□

Spring 2024 #5. For each $n \in \mathbb{N}$, let $f_n : [0,1] \to \mathbb{R}$ be defined by $f_n(x) = x^n$. Find the norm of f_n in the following spaces:

(a) $C([0,1])$, with the norm $\|f\|_{C([0,1])} = \sup_{x\in[0,1]} |f(x)|$.

Solution. It can be shown (by using the derivative) that the functions f_n are non-negative and increasing on $[0,1]$. Thus, we have that

$$\|f_n\| = \sup_{x\in[0,1]} |f(x)| = f(1) = 1$$

for all $n \in \mathbb{N}$. □

(b) $L^1([0,1])$, with the standard L^1 norm.

Solution. Recall that

$$\|f_n\|_{L^1} = \int_{[0,1]} |f_n(x)|\ dm,$$

where m is Lebesgue measure. However, the functions f_n are continuous and non-negative in $[0,1]$. Thus, the Lebesgue integral coincides with the Riemann integral, so that

$$\begin{aligned}\|f_n\|_{L^1} &= \int_{[0,1]} |f_n(x)|\ dm \\ &= \int_0^1 x^n\ dx \\ &= \frac{1}{n+1}.\end{aligned}$$

□

Spring 2024 #6. Let X be a Banach space. (Recall that this means that X is a normed vector space that is complete with respect to the metric induced by that norm.) Let S be a closed linear subspace of X. Prove that S is a Banach space. (For the norm on S, take the restriction to S of the norm on X.)

Proof: The restriction of a norm to a subspace is a norm on that subspace.

It remains to show that the metric on S induced by that norm is complete.

Let $\{x_n\}$ be a Cauchy sequence in S. We will show that there exists $x \in S$ such that $x_n \to x$.

Because X is complete, we know that there exists $x \in X$ such that $x_n \to x$.

By def. of closed, we have that $x \in S$.

Therefore S is complete, because every Cauchy sequence in S converges to a point in S.

Therefore S is a Banach space.

Spring 2024 #7. Let X be an inner product space over $\mathbf{R}$. Show that two vectors $x, y \in X$ are orthogonal if and only if

$$\|x + \alpha y\| = \|x - \alpha y\|$$

for every $\alpha \in \mathbf{R}$.

Proof. ($\Rightarrow$) Assume x, y are orthogonal, and let $\langle x, y\rangle$ denote their inner product. Then $\langle x, y\rangle = 0$. Let $\alpha \in \mathbf{F}$ be arbitrary. Then

$$\begin{aligned} \|x + \alpha y\|^2 &= \langle x + \alpha y, x + \alpha y\rangle \\ &= \langle x, x\rangle + \alpha\langle y, x\rangle + \alpha\langle x, y\rangle + \alpha^2\langle y, y\rangle \\ &= \langle x, x\rangle + \alpha^2\langle y, y\rangle. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \|x - \alpha y\|^2 &= \langle x - \alpha y, x - \alpha y\rangle \\ &= \langle x, x\rangle - \alpha\langle y, x\rangle - \alpha\langle x, y\rangle + \alpha^2\langle y, y\rangle \\ &= \langle x, x\rangle + \alpha^2\langle y, y\rangle. \end{aligned}$$

Thus $\|x + \alpha y\| = \|x - \alpha y\|$.

($\Leftarrow$) Assume now that

$$\|x + \alpha y\| = \|x - \alpha y\|$$

for all $\alpha \in \mathbb{R}$. Then if we expand the norms in terms of the inner products, we have

$$\|x + \alpha y\|^2 = \langle x, x\rangle + \alpha\langle y, x\rangle + \alpha\langle x, y\rangle + \alpha^2\langle y, y\rangle$$

and

$$\|x - \alpha y\|^2 = \langle x, x\rangle - \alpha\langle y, x\rangle - \alpha\langle x, y\rangle + \alpha^2\langle y, y\rangle.$$

By assumption, these are equal. Thus, we see that

$$\langle x, x\rangle + \alpha\langle y, x\rangle + \alpha\langle x, y\rangle + \alpha^2\langle y, y\rangle = \langle x, x\rangle - \alpha\langle y, x\rangle - \alpha\langle x, y\rangle + \alpha^2\langle y, y\rangle,$$

which reduces to

$$\alpha\langle y, x\rangle + \alpha\langle x, y\rangle = -\alpha\langle y, x\rangle - \alpha\langle x, y\rangle.$$

Since $\langle x, y\rangle = \langle y, x\rangle$, this equation simplifies to

$$4\alpha\langle x, y\rangle = 0.$$

Since this holds for every α, we may choose $\alpha \neq 0$, in which case we obtain that

$$\langle x, y\rangle = 0.$$

It follows that x, y are orthogonal. $\square$