Algebra Comprehensive Exam Fall 2021, new style

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Answer five (5) questions only. You must answer *at least one* from each of section: (I) Linear algebra, (II) Group theory, and (III) Synthesis: linear algebra and group theory. Indicate CLEARLY which problems you want us to grade; otherwise, we will select the first problem from each section, and then the first two additional problems answered after that. Be sure to show enough work that your answers are adequately supported. Tip: When a question has multiple parts, the later parts often (but not always) make use of the earlier parts.

Notation: Unless otherwise stated, $\mathbb{N}$, $\mathbb{Q}, \mathbb{Z}, \mathbb{Z}_n, \mathbb{C}$, and $\mathbb{R}$ denote the sets of natural numbers, rational numbers, integers, integers modulo n, complex numbers, and real numbers respectively, regarded as groups or fields or vector spaces in the usual way.

Linear algebra

(L1) Let V and W be vector spaces over a field F. Let 0_V and 0_W be the zero vectors of V and W respectively. Let $T : V \to W$ be a linear transformation. Let $N(T)$ denote the nullspace of T.

(a) Show that $T(0_V) = 0_W$.

(b) Show that T is one-to-one if and only if $N(T) = \{0_V\}$.

Solution: *(a) We have that $T(0_V) = T(0_V + 0_V) = T(0_V) + T(0_V)$. Now add $-T(0_V)$ to both sides to get that $0_W = T(0_V)$. (b) Suppose T is one-to-one. We know from part a that 0_V is in the nullspace of T. Suppose x is in the nullspace. Then $T(x) = 0_W = T(0_V)$. Since T is one-to-one we have that $x = 0_V$. Thus, $N(T) = \{0_V\}$. Conversely, suppose that $N(T) = \{0_V\}$. And suppose $T(x) = T(y)$. Then $T(x) - T(y) = 0_W$. So $T(x - y) = 0_W$. Thus $x - y \in N(T)$. So $x - y = 0_V$. Thus $x = y$. So T is one-to-one.*

(L2) Let V_1 and V_2 be proper subspaces of a vector space V. Show that $V_1 \cup V_2$ is a proper subset of V.

(Recall that A is a proper subset of B if A is a subset of B but not equal to B.)

Solution: *Suppose, to the contrary, that $V_1 \cup V_2 = V$. Since V_1 and V_2 are proper subsets of V, we can't have $V_1 \subseteq V_2$ or $V_2 \subseteq V_1$, so there are vectors $x_1 \in V_1 \setminus V_2$ and $x_2 \in V_2 \setminus V_1$. Since $x_1 + x_2 \in V = V_1 \cup V_2$, we have two cases: If $x_1 + x_2 \in V_1$, then $x_2 \in V_1$, a contradiction. If $x_1 + x_2 \in V_2$, then $x_1 \in V_2$, a contradiction.*

(L3) Let V be the space of real functions spanned by $B = \{\sin^2 x, \cos^2 x, \sin x \cos x\}$. Let $\phi : V \to \mathbb{R}^3$ be the linear map defined by $\phi(f) = (f(0), f'(0), f''(0))$ for all functions $f \in V$. Show that ϕ is an isomorphism.

<u>Solution:</u> *Because differentiation and evaluations are linear maps from real functions to real functions, ϕ is linear. By a calculation we have*

$$(*) \qquad \phi(c_1 \sin^2 x + c_2 \cos^2 x + c_3 \sin x \cos x) = (c_2, c_3, 2(c_1 - c_2))$$

for all $c_1, c_2, c_3 \in \mathbb{R}$. Define $\psi : \mathbb{R}^3 \to V$ by

$$\psi(x_1, x_2, x_3) = \frac{1}{2}(2x_1 + x_3) \sin^2 x + x_1 \cos^2 x + x_2 \sin x \cos x$$

for all $x_1, x_2, x_3 \in \mathbb{R}$. It is now easy to check that $\phi \circ \psi = \mathrm{id}_{\mathbb{R}^3}$ and $\psi \circ \phi = \mathrm{id}_V$, so these functions are inverse bijections. In particular, ϕ is an isomorphism.

- OR-

Because differentiation and evaluations are linear maps from real functions to real functions, ϕ is linear. ϕ is surjective because, for any $(x_1, x_2, x_3) \in \mathbb{R}^3$, we have

$$(x_1, x_2, x_3) = \phi(1/2(2x_1 + x_3) \sin^2 x + x_1 \cos^2 x + x_2 \sin x \cos x).$$

Hence $\dim(\operatorname{im} \phi) = \dim(\mathbb{R}^3) = 3$. Also, because V has a three element spanning set, B, we have $\dim(V) \leq 3$. Because $\dim V = \dim(\operatorname{im} \phi) + \dim(\ker \phi)$, this can happen only if $\dim V = 3$ and $\dim(\ker \phi) = 0$, in particular, $\ker \phi = \{0\}$ and ϕ is injective. Since ϕ is both surjective and injective, ϕ is an isomorphism.

- OR-

Because differentiation and evaluations are linear maps from real functions to real functions, ϕ is linear. We show that B is linearly independent: Suppose that

$$c_1 \sin^2 x + c_2 \cos^2 x + c_3 \sin x \cos x = 0$$

for some $c_1, c_2, c_3 \in \mathbb{R}$. Evaluating this at $x = 0$, $x = \pi/4$ and $x = \pi/2$ gives the equations $c_2 = 0$, $c_1 + c_2 + c_3 = 0$ and $c_1 = 0$, respectively. These equations imply $c_1 = c_2 = c_3 = 0$. This means that B is linearly independent. Since B also spans V, it is a basis for V.
From (*), we see that ϕ is represented by the matrix

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -2 & 0 \end{bmatrix}$$

with respect to the basis B of V and the standard basis of R^3. This matrix has determinant 2 so is invertible. That means that ϕ is also invertible, and hence an isomorphism.

Groups

(G1) Let G be an abelian group. Show that $H = \{x \in G \mid |x| \text{ is finite}\}$ is a subgroup of G. (Note: Here $|x|$ denotes the order of x.)

<u>Solution:</u> *Reminder: $|x|$ is finite if and only if $x^n = e$ for some n.*

(1) H closed under the group operation*: Let $x, y \in H$. Then $x^m = y^n = e$ for some $m, n \in \mathbb{Z}$, so using associativity and commutativity, $(xy)^{mn} = (x^m)^n (y^n)^m = e^n e^m = e$, hence $xy \in H$.*

(2) $e \in H$*: Obvious, since* $|e| = 1$.

(3) H closed under taking inverses*: Let* $x \in H$. *Then* $x^n = e$ *for some* n *and so* $x^{-1} = x^{n-1}$. *Hence* $(x^{-1})^n = (x^{n-1})^n = x^{n(n-1)} = (x^n)^{n-1} = e^{n-1} = e$ *and* $x^{-1} \in H$.

OR

Since $\langle x \rangle = \langle x^{-1} \rangle$, *we have* $|x^{-1}| = |x|$, *from which the claim is clear.*

(G2) The **center** of a group G is defined as

$$Z(G) = \{g \in G : gx = xg \text{ for all } x \in G\}.$$

(a) Prove $Z(G)$ is a normal subgroup of G.
(b) Prove: If $G/Z(G)$ is cyclic, then G is abelian.

<u>Solution:</u> *For problem (a) see here: https://en.wikipedia.org/wiki/Center_(group_theory) and also problem 2 from here: http://pi.math.cornell.edu/~riley/Teaching/Groups_and_Geometry2012/past_exams/2011prelim2_with_solutions.pdf. For (b) see problem 11 from here on page 2: https://www.math.utah.edu/~schwede/math435/HW4Sols.pdf*

(G3) *Let* G *be a group and* $k \in \mathbb{N}$. *Prove: If* H *is the only subgroup of* G *with order* k, *then* H *is a normal subgroup of* G.

<u>Solution:</u> *Suppose that* H *is the only subgroup of* G *with order* k. *Let* $g \in G$. *Define* $\phi_g : G \to G$ *by* $\phi_g(x) = g^{-1}xg$. *First show that* ϕ_g *is an isomorphism. Then* $\phi_g(H) = g^{-1}Hg$ *will be a subgroup of* G *of the same size as* H. *Thus,* $g^{-1}Hg = H$. *Since this is true for all* $g \in G$ *we have that* H *is a normal subgroup of* G.

Synthesis: Linear algebra and group theory

(S1) *Let* V *be a vector space over a field* F. *Let*

$$G = \{T : V \to V \mid T \textit{ is a linear transformation } \}$$

(a) Show that G *is a group under function addition. That is, the group operation is defined to be* $(T_1 + T_2)(x) = T_1(x) + T_2(x)$ *when* $T_1, T_2 \in G$.

(b) Let

$$H = \{T \in G \mid \textit{ there exists some } \alpha \textit{ where } T(x) = \alpha x \textit{ for all } x \textit{ in } V\}$$

Show that H *is a subgroup of* G.

<u>Solution:</u> *(a) The identity element is the zero function* $T_0(x) = 0$ *for all* $x \in V$. *Let* $T_1, T_2, T_3 \in G$. *Given* $x \in V$ *we have that* $((T_1 + T_2) + T_3)(x) = (T_1(x) + T_2(x)) + T_3(x) = T_1(x) + (T_2(x) + T_3(x)) = (T_1 + (T_2 + T_3))(x)$ *which gives associativity. We have that* $T_1 + T_2 \in G$ *since* $(T_1 + T_2)(\alpha x_1 + \beta x_2) = T_1(\alpha x_1 + \beta x_2) + T_2(\alpha x_1 + \beta x_2) = \alpha T_1(x_1) + \beta T_1(x_2) + \alpha T_2(x_1) + \beta T_2(x_2) = \alpha(T_1 + T_2)(x_1) + \beta(T_1 + T_2)(x_2)$. *So* G *is closed under function addition. We have that* $-T_1$ *is also a linear transformation since* $-T_1(\alpha x_1 + \beta x_2) =$

$-\alpha T_1(x_1) - \beta T_1(x_2) = \alpha(-T_1(x_1)) + \beta(-T_1(x_2))$. *So* $-T_1 \in G$ *and* G *is closed under inversion. Thus* G *is a group.*

(b) First note that if $T : V \to V$ *is defined by* $T(x) = \alpha x$ *then* T *is a linear transformation because* $T(c_1v_1 + c_2v_2) = \alpha(c_1v_1 + c_2v_2) = c_1\alpha v_1 + c_2\alpha v_2 = c_1T(v_1) + c_2T(v_2)$.

The zero function T_0 *is in* H *since* $T_0(x) = 0 \cdot x$ *for all* $x \in V$. *Let* $T_1, T_2 \in H$. *Then* $T_1(x) = \alpha x$ *and* $T_2(x) = \beta x$ *for all* $x \in V$. *Thus,* $(T_1+T_2)(x) = T_1(x)+T_2(x) = \alpha x+\beta x = (\alpha + \beta)x$. *So* $T_1 + T_2 \in H$. *Also,* $-T_1(x) = (-\alpha)x$ *for all* $x \in V$. *Thus* $-T_1 \in H$. *So* H *is a subgroup of* G.

(S2)

(a) *Let* $GL_2(\mathbb{R})$ *and* $GL_2(\mathbb{C})$ *be the groups of* 2×2 *invertible matrices with entries in* $\mathbb{R}$ *and* $\mathbb{C}$ *respectively. Clearly,* $GL_2(\mathbb{R})$ *is a subgroup of* $GL_2(\mathbb{C})$. *Is* $GL_2(\mathbb{R})$ *is a* ***normal*** *subgroup of* $GL_2(\mathbb{C})$? *Proof or counterexample please.*

(b) *Let* $GL_n(\mathbb{R})$ *be the group of* $n \times n$ *invertible matrices with entries in* $\mathbb{R}$. *Let* $SL_n(\mathbb{R})$ *be the group of* $n \times n$ *matrices with determinant 1 and entries in* $\mathbb{R}$. *Prove or disprove:* $SL_n(\mathbb{R})$ *is a normal subgroup of* $GL_n(\mathbb{R})$.

Solution:

(a) $GL_2(\mathbb{R})$ *is not a normal subgroup of* $GL_2(\mathbb{C})$. *Counterexample: If* $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in GL_2(\mathbb{R})$ *and* $B = \begin{bmatrix} i & 0 \\ 0 & 1 \end{bmatrix} \in GL_2(\mathbb{C})$, *then* $B^{-1}AB = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \notin GL_2(\mathbb{R})$.

(b) *Let* $A \in SL_n(\mathbb{R})$ *and* $B \in GL_n(\mathbb{R})$. *Then* $\det(BAB^{-1}) = (\det B)(\det A)(\det B^{-1}) = \det A = 1$. *Hence,* $BAB^{-1} \in SL_n(\mathbb{R})$. *Therefore,* $SL_n(\mathbb{R})$ *is a normal subgroup of* $GL_n(\mathbb{R})$.

(S3) *Let* $SO_2(\mathbb{R})$ *be the group of* 2×2 *orthonormal matrices (that is,* $A^{-1} = A^T$*) with determinant 1 and real entries. Let* $U(1) = \{z \in \mathbb{C} : |z| = 1\}$. *Prove:* $SO_2(\mathbb{R}) \cong U(1)$.

Solution: *Observe, if* $A \in SO_2(\mathbb{R})$ *then* $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ *and* $a^2 + b^2 = 1$.

Define $\phi : SO_2(\mathbb{R}) \to U(1)$ *by*

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \mapsto a + bi.$$

One can show that ϕ *is an isomorphism.*
